What are we trying to accomplish?

We would like to understand the geometric conditions that are imposed upon a non-atomic measure $\mu$ from the $L^2(\mu)$ boundedness of an associated Calderón-Zygmund operator.
Fix $s \in (0, d)$. A Calderón-Zygmund kernel of dimension $s$ is an odd function $K : \mathbb{R}^d \setminus \{0\} \to \mathbb{R}^d$ satisfying

$$|K(x)| \leq \frac{1}{|x|^s} \text{ and } |
abla K(x)| \leq \frac{1}{|x|^{s+1}}$$

for every $x \in \mathbb{R}^d \setminus \{0\}$. 

Notation

We say that a CZO (with CZ kernel $K$) is bounded in $L^2(\mu)$ if

$$\sup_{\varepsilon > 0} \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d \setminus B(x, \varepsilon)} K(x-y)f(y) \, d\mu(y) \right|^2 \, d\mu(x) \leq C \|f\|^2_{L^2(\mu)}$$

for every $f \in L^2(\mu)$. 

Ben Jaye (Kent State) 
Reflectionless Measures 
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for every $f \in L^2(\mu)$. 
What would we like to know about $\mu$?

– If $s \in \mathbb{Z}$, then we would like to determine whether $\mu$ is supported in some collection of Lipschitz surfaces (assuming that supp($\mu$) has dimension $s$). (Jones '89, David-Semmes '91, '93, Mattila-Melnikov-Verdera '96, David-Mattila '98, David-Leger '99, Nazarov-Tolsa-Volberg '12, Hofmann-Martel-Mayboroda-Uriate-Tuero '12).

– If the CZO has non-integer dimension, then we would like to know sharp conditions on the density function of the measure. (Mateu-Prat-Verdera '05, Tolsa '11, Eiderman-Nazarov-Volberg '11, Reguera-Tolsa '14.)

Theorem (Jaye-Nazarov-Reguera-Tolsa, '16)
Fix $s \in (d-1, d)$. Suppose that the $s$-Riesz transform (the CZO with kernel $K(x) = |x|^{s+1}$) is bounded in $L^2(\mu)$, then there is a constant $C > 0$ such that
\[
\int_Q \int_0^{\infty} \left( \mu(B(x, r) \cap Q) r^s \right)^2 dr d\mu(x) \leq C \mu(Q)
\]
for every cube $Q \subset \mathbb{R}^d$. 
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**Theorem (Jaye-Nazarov-Reguera-Tolsa, ’16)**

Fix \( s \in (d-1, d) \). Suppose that the \( s \)-Riesz transform (the CZO with kernel \( K(x) = \frac{x}{|x|^{s+1}} \)) is bounded in \( L^2(\mu) \), then there is a constant \( C > 0 \) such that

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for every cube \( Q \subset \mathbb{R}^d \).
What is a reflectionless measure (associated to a integral kernel $K$)?

– It is a measure for which the potential

$$\int_{\mathbb{R}^d} K(x - y) d\mu(y)$$

is constant for $x$ on the support of $\mu$ (when considered in a suitable weak sense).

Examples

– The Hausdorff measure of a $k$-plane is a reflectionless measure for any CZO.

– Periodic configurations of $k$-planes is also reflectionless.
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More Information

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– We shall present a result in the opposite direction.
Consider the kernel \( K(z) = \frac{1}{|z|} \left( \frac{\bar{z}}{|z|} \right)^3 \).

Then the 2 dimensional Lebesgue measure restricted to a ball \( B(z_0, r) \) is reflectionless in the sense that
\[
\int_{B(z_0, r)} K(z - \omega) \, dm(\omega) = 0 \quad \text{for all} \quad z \in B(z_0, r).
\]

There is a one dimensional purely unrectifiable measure \( \mu \) for which the singular integral operator associated to the kernel \( K(z) = \bar{z} z^2 \) is bounded in \( L^2(\mu) \).

Compare to David-Leger: If the Cauchy transform of a 1-dimensional non-atomic measure \( \mu \) is bounded in \( L^2(\mu) \), then the support of \( \mu \) is rectifiable. This result was generalized by Chousionis, Mateu, Prat, Tolsa to other kernels.
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Construction of the measure

Take very fast decaying sequence \((r_n)_n\). First put \(1/r_1\) roughly equally spaces discs \(D^{(1)}_k\) of radius \(r_1\) in \(B(0, 1)\). Then put \(r_1/r_2\) roughly equally spaced discs \(D^{(2)}_k\) of radius \(r_2\) in each of the discs of radius \(r_1\). Continue in this manner......
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Pass to a subsequence with a limit. Get a limit (probability) measure $\mu$, supported on the Cantor dust (let's call it $K$).

We show that, for every generation $n$, and $z \in \text{supp}(\mu)$,

$$\left| \int D_n(z) \setminus D_{n+1}(z) K(z - \xi) \, d\mu(\xi) \right| \leq \sqrt{r_{n+1} r_n}.$$ 

Then the $T(1)$-theorem ensures that the CZO associated to $K$ is bounded in $L^2(\mu)$, provided that

$$\sum_{n=1}^{\infty} \sqrt{r_{n+1} r_n} < \infty.$$ 

For our dust $K$, $H^1(K \cap \Gamma) = 0$ for any rectifiable curve $\Gamma$ (just density considerations).
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**Remark 1.** For the Cantor dust measure \( \mu \), the limit

\[
\lim_{\varepsilon \to 0} \int_{\mathbb{C} \setminus B(z,r)} \frac{z - \omega}{(z - \omega)^2} d\mu(\omega)
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**Remarks!**

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**Remarks!**

**Open Problem 1.** Does there exist an AD-regular measure $\mu$ (this should satisfy, for some constant $C > 0$, $\frac{1}{C} r \leq \mu(B(x, r)) \leq C r$ for all $x \in \text{supp}(\mu)$ and small $r > 0$) supported on an unrectifiable set $K$ for which the three revolutions singular integral operator is bounded in $L^2(\mu)$?
The End