Superstring scattering amplitudes and modular forms

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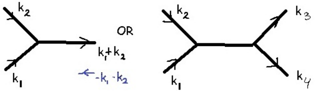
Quantum field theory

- *Particle*: point traveling in space-time
- *Trajectory*: 

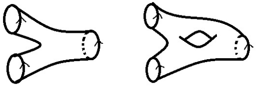
- *Interaction*:


so a singular trajectory
- *Probability amplitude*:
integral over all possible trajectories (have a propagator for each free movement, and probabilities for each interaction)
- *Problem:*
the integral for many (> 4) interactions diverges

String theory

- *Particle*: string = a small circle
- *Trajectory*: 

- *Interaction*:



- *Probability amplitude*:
  integral over all possible trajectories, i.e. over all possible surfaces with given end-circles.
- *Question*:
  how to assign a weight to a given surface, i.e. what is the probability distribution on the set of all trajectories?

Bosonic string, Ramond-Neveu-Schwarz formalism

$k_1, \dots, k_n$	the momenta of incoming/outgoing particles
X	Riemann surface of genus g with n circles as ends
M	26-dimensional manifold with a Riemannian metric
$\phi : X \to M$	a worldsheet (trajectory of a string)

We integrate over the space of all such maps: amplitude

$$A_g(k_1, \dots, k_n) = \int_{X,\phi} e^{-I(X,\phi)} \prod_{i=1}^{n} V(k_i, X, \phi) D(X, \phi)$$

The total probability is then

$$A(k_1, \dots, k_n) = \sum_{g=0}^{\infty} \lambda^{2g-2} A_g(k_1, \dots, k_n)$$

Simplest case: no incoming or outgoing particles.
Free energy = vacuum-to-vacuum probability.

Bosonic measure, physics formulation

$$A_g^{bos} := \int\limits_{X,\,\phi: X \to M} e^{-I(X,\phi)} D(X,\phi)$$

What is the action I (and the measure D)?

z	holomorphic coordinate on X
$h_{a,b}(z)$	metric on X
$x^\mu(z)$	the coordinates on $M^{26} \supset \phi(X)$
$s_{\mu\nu}(x)$	Riemannian metric on M^{26}

$$I(X,\phi) := \int_X dz\, d\overline{z}\, \sqrt{\det h(z)}\, h^{ab}(z)\, \partial_a x^\mu(z)\, \partial_b x^\nu(z) s_{\mu\nu}(x(z))$$

The action is invariant under conformal transformations. Thus the integrand in A_g only depends on the complex structure on X, not the map or the metric on M^{26}.
(*This is a physical argument: reducing an infinite-dimensional integral over all worldsheets to a finite-dimensional one*)

Bosonic measure, mathematics formulation

$$A_g^{bos} = \int_{\mathcal{M}_g} ||d\mu_{\mathrm{bos}}||^2$$

$d\mu_{bos}$ is a top degree holomorphic form on $\mathcal{M}_g$, i.e. a $(3g-3,0)$ form, i.e. a canonical form (section of the canonical bundle $K_{\mathcal{M}_g}$).

Genus 0: $\mathcal{M}_0 =$one point, there is nothing to integrate.
(*An interesting expression for the 4-point function* $A_0(k_1,k_2,k_3,k_4)$)

In genus 1 we have

$$A_1^{bos} = \int_{\mathcal{M}_1 = \mathbb{H}/SL(2,\mathbb{Z})} \frac{1}{(\operatorname{Im}\tau)^{14}} \left| \frac{d\tau}{\prod \theta_m^8(\tau)} \right|^2$$

(*Explicit expressions also known for* $g = 2,3$)

Problem:
The bosonic measure is not integrable: it blows up near $\partial\overline{\mathcal{M}_g}$ as $\frac{dz}{z^2}$, and thus $\int_{\mathcal{M}_g} ||d\mu_{bos}^2|| = \infty$

Superstrings, Ramond-Neveu-Schwarz formalism

X	a Riemann surface of genus g with n circles as ends
M	10-dimensional supermanifold with a Riemannian metric
$\phi : X \to M$	a worldsheet (trajectory of a string)

$$A_g^{ss} := \int_{X,\,\phi:X\to M} e^{-I^{ss}(X,\phi)} D(X,\phi)$$

Even fields:

coordinates x^μ on M^{10}, metric $s_{\mu\nu}$ on M

Odd fields:

coordinates $\psi_\pm(z)(dz)^{1/2}$; gravitino $\begin{matrix}\chi_z^-(z)\\ \chi_{\bar z}^+(z)\end{matrix}(d\bar z)\otimes(dz)^{1/2}$.

$$I^{ss}(X,\phi) := \frac{1}{4\pi}\int_X dz\, d\bar z s_{\mu\nu}\Big(\partial_z x^\mu \partial_{\bar z} x^\nu - \psi_+^\mu \partial_{\bar z}\psi_+^\nu$$

$$-\psi_-^\mu \partial_z \psi_-^\nu - \frac{1}{2}\chi_{\bar z}^+ \chi_z^- \psi_+^\mu \psi_-^\nu + \chi_{\bar z}^+ \psi_+^\mu \partial_z x^\nu + \chi_z^- \psi_-^\mu \partial_{\bar z} x^\nu\Big)$$

Superstring measure

Physics prediction: superconformal invariance,...;
⇒everything should depend only on the superRiemann surface, so

$$A_g^{ss} = \int_{s\mathcal{M}_g} ||d\mu_{ss}||^2$$

$$\dim s\mathcal{M}_g = (3g-3; 2g-2)$$

Gauge-fixing: choose a section $s\mathcal{M}_g \to$space of maps (X,ϕ)
Difficulties: (for $g > 1$)

— The standard (Faddeev-Popov) gauge-fixing does not work for supermoduli. Have to do chiral splitting first.

— Impossible to choose a holomoprhic section and preserve supersymmetry. Thus need to deform the complex structure simultaneously with other coordinates.

D'Hoker-Phong: successfully dealt with this for $g = 2$.

Known results

- The bosonic measure is known explicitly for $g \leq 4$; for $g > 4$: known in terms of extra points on the Riemann surface, or of Weil-Petersson volume, ... (Beilinson, Belavin, D'Hoker, Knizhnik, Manin, Morozov, Phong, Verlinde, Verlinde)
- Superstring measure known for $g = 1$ (Green, Schwarz; Gross, Harvey, Martinec, Rohm 1980's).
- Supermoduli difficulties overcome for $g = 2$ (D'Hoker, Phong breakthrough, 2000's).
- *Goal for today:* a viable ansatz for the superstring measure in terms of the bosonic measure, for $g \leq 5$

Ansatz (G.)

$$d\mu_{ss}[m](\tau) = d\mu_{bos} \sum_{i=0}^{g} (-1)^i 2^{i(i-1)/2} \sum_{V \subset (\mathbb{Z}/2)^{2g}; \dim V = i} \prod_{n \in V} \theta_{n+m}^{2^{4-i}}$$

Mathematics notation: modular forms

$\mathcal{H}_g :=$ Siegel upper half-space of dimension g
$= \text{set of period matrices } \{\tau \in Mat_{g\times g}(\mathbb{C}) \mid \tau^t = \tau,\ \operatorname{Im}\tau > 0\}$

The action of $\mathrm{Sp}(2g, \mathbb{Z})$ on $\mathcal{H}_g$ is given by

$$\gamma \circ \tau := (C\tau + D)^{-1}(A\tau + B)$$

for an element $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(2g, \mathbb{Z})$

Definition

A modular form of weight k with respect to $\Gamma \subset \mathrm{Sp}(2g, \mathbb{Z})$ is a function $F : \mathcal{H}_g \to \mathbb{C}$ such that

$$F(\gamma \circ \tau) = \det(C\tau + D)^k F(\tau) \qquad \forall \gamma \in \Gamma, \forall \tau \in \mathcal{H}_g$$

Mathematics notation: theta functions and constants

For $\tau \in \mathcal{H}_g$ let $A_\tau := \mathbb{C}^g/\mathbb{Z}^g + \tau\mathbb{Z}^g$ be the abelian variety.

The theta function of $\tau \in \mathcal{H}_g,\ z \in \mathbb{C}^g$ is

$$\theta(\tau, z) := \sum_{n \in \mathbb{Z}^g} e^{\pi i (n^t(\tau n + 2z))}$$

The values of the theta function at points of order two $m := (\tau a + b)/2$ for $a, b \in (\mathbb{Z}/2)^g$)

$$\theta_m(\tau) := \theta \begin{bmatrix} a \\ b \end{bmatrix} (\tau) := \theta(\tau, \frac{\tau a + b}{2}) e^{\cdots}$$

are modular forms of weight one half *(for a finite index subgroup $\Gamma(4,8) \subset \mathrm{Sp}(2g, \mathbb{Z})$)*, called theta constants.

If m is odd, i.e. $a \cdot b = 1 \in \mathbb{Z}/2$, then $\theta_m(\tau) \equiv 0$.

Moduli of abelian varieties

- Abelian variety A: a projective variety with a group structure $\Longleftrightarrow A_\tau = \mathbb{C}^g/\mathbb{Z}^g + \mathbb{Z}^g\tau$.
- Principal polarization Θ on A: an ample divisor on A with $h^0(A, \Theta) = 1$. $\Longleftrightarrow \Theta_\tau = \{z \in A_\tau | \theta(\tau, z) = 0\}$.
  (*Theta function is a section of a line bundle on A_τ, i.e. $\forall a, b \in \mathbb{Z}^g$ we have $\theta(\tau, z + \tau a + b) = \exp(\cdot)\theta(\tau, z)$*)
- $\mathcal{A}_g$: moduli space of principally polarized abelian varieties.
- Then $\mathcal{H}_g \to \mathcal{A}_g$ (sending τ to (A_τ, Θ_τ)) is the universal cover, and $\mathcal{A}_g = \mathcal{H}_g / \mathrm{Sp}(2g, \mathbb{Z})$.

Jacobians of Riemann surfaces = curves

$\mathcal{M}_g$:=moduli of curves/Riemann surfaces of genus g
For $X \in \mathcal{M}_g$ the Jacobian $Jac(X) := \mathrm{Pic}^{g-1}(X) \in \mathcal{A}_g$.
Equivalently for a basis $A_1, \dots, A_g, B_1, \dots, B_g$ of $H_1(X, \mathbb{Z})$, and a basis $\omega_1, \dots, \omega_g \in H^{1,0}(X, \mathbb{C})$ with $\int_{A_i} \omega_j = \delta_{i,j}$ let $\tau_{ij} := \int_{B_i} \omega_j$.

Torelli theorem

The map $X \to Jac(X)$ is an embedding $\mathcal{M}_g \hookrightarrow \mathcal{A}_g$.

The image is called the Jacobian locus $\mathcal{J}_g \subset \mathcal{A}_g$.

Hodge bundle L:= the line bundle of modular forms of weight 1 on $\mathcal{A}_g$, and its restriction to $\mathcal{M}_g$

Theorem (Mumford)

$K_{\mathcal{M}_g} = L^{\otimes 13}$

Thus to construct $d\mu_{bos}$ could try to write a modular form of weight 13 on $\mathcal{A}_g$, and restrict it to $\mathcal{M}_g$.

Bosonic measure in low genus (Belavin, Knizhnik)

$$A_1^{bos} = \int_{\mathcal{M}_1} |d\tau|^2 (\mathrm{Im}\,\tau)^{-14} \prod_{m \text{ even}} |\theta_m^{-8}(\tau)|^2$$

$$A_2^{bos} = \int_{\mathcal{M}_2} \prod_{1 \le i \le j \le 2} |d\tau_{ij}|^2 (\det \mathrm{Im}\,\tau)^{-13} \prod_{m \text{ even}} |\theta_m^{-2}(\tau)|^2$$

$$A_3^{bos} = \int_{\mathcal{M}_3} \prod_{1 \le i \le j \le 3} |d\tau_{ij}|^2 (\det \mathrm{Im}\,\tau)^{-13} \prod_{m \text{ even}} |\theta_m^{-\frac{1}{2}}(\tau)|^2$$

No reason to expect such formulas for $g \ge 4$, when $\mathcal{M}_g \subsetneq \mathcal{A}_g$.

g	$\dim \mathcal{M}_g$		$\dim \mathcal{A}_g$	
1	1	$=$	1	
2	3	$=$	3	$\mathcal{M}_g = \mathcal{A}_g^{indecomposable}$
3	6	$=$	6	
4	9	$+1 =$	10	Schottky's equation for $\mathcal{J}_4 \subset \mathcal{A}_4$
g	$3g-3$	$+\frac{(g-3)(g-2)}{2} =$	$\frac{g(g+1)}{2}$	partial results

Superstring measure

$$A_g^{ss} = \int_{s\mathcal{M}_g} ||d\mu_{s\mathcal{M}_g}||^2$$

Expect to have the pullback of $d\mu_{bos}$ from $\mathcal{M}_g$ to $s\mathcal{M}_g$ as a factor in $d\mu_{s\mathcal{M}_g}$.
Need to integrate out the odd supermoduli.
Physics prediction: invariance under the superconformal group, so integrate over a section of $s\mathcal{M}_g \to$space of maps (X, ϕ).
Difficulties: the standard gauge-fixing methods do not work; cannot choose a holomorphic section preserving supersymmetry.
Mathematics approach: construct a finite cover $\mathcal{S}_g \to \mathcal{M}_g$ such that $s\mathcal{M}_g \to \mathcal{S}_g$

Spin curves

What is $\psi_\pm(z)(dz)^{1/2}$? It is a section of $K_X^{\otimes(1/2)}$.

Difficulty: there are many square roots. Can add any point of order two on $Jac(X)$, so there are 2^{2g} different square roots $\eta^{\otimes 2} = K_X$.

Let $\mathcal{S}_g := \{$moduli of pairs $(X, \eta)\}$; $\mathcal{S}_g \to \mathcal{M}_g$ is a $2^{2g} : 1$ cover.

Integrating out the odd moduli gives the superstring measure as a function on $\mathcal{S}_g$.

Depending on whether $h^0(X, \eta)$ is even or odd (generically 0 or 1), have two irreducible components $\mathcal{S}_g = \mathcal{S}_g^+ \sqcup \mathcal{S}_g^-$.

For supersymmetry reasons, the measure on $\mathcal{S}_g^-$ (where η has a non-trivial section) is identically zero.

Mathematics statement: the superstring measure

$$A_g^{ss} = \int_{\mathcal{S}_g^+} ||d\mu_{ss}||^2$$

Expect to have a product

$$d\mu_{ss}(\tau,\eta) = (\det \operatorname{Im}\tau)^{-8}\, \Xi[m](\tau) d\mu_{bos}(\tau)$$

for Ξ holomorphic modular form of weight 8.

$\Xi^{(1)}[m](\tau) = \theta_m^4(\tau) \prod_{\text{all three even } n} \theta_n^4(\tau)$ (Green-Schwarz)

Theorem (D'Hoker-Phong)

$$\Xi^{(2)}[m](\tau) = \theta_m^4(\tau) \cdot \sum_{1\le i\le j\le 3} (-1)^{\nu_i\cdot\nu_j} \prod_{k=4,5,6} \theta[\nu_i+\nu_j+\nu_k]^4(\tau)$$

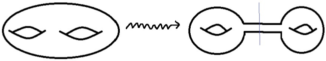
where $\nu_1,\ldots,\nu_6$ are the 6 odd spin structures such that $m = \nu_1+\nu_2+\nu_3$.

(*It is natural to expect the $\theta_m^4(\tau)$ factor to have 0-3 point functions vanishing.*)

This is very hard, uses heavily the hyperelliptic representation of genus 2 surfaces. So how one can generalize to higher genus?

D'Hoker-Phong program: factorization constraint

Study the degeneration of $d\mu_{ss}$ and $d\mu_{bos}$ under degeneration



Expect no transfer of momentum over the long cylinder.
Both $d\mu_{ss}$ and $d\mu_{bos}$ become infinite in the limit, extract the lowest order term.

Factorization constraint (D'Hoker-Phong)

$$\Xi^{(g)}[m]\begin{pmatrix}\tau_1 & 0\\ 0 & \tau_2\end{pmatrix} = \Xi^{(g_1)}[m_1](\tau_1)\cdot \Xi^{(g_2)}[m_2](\tau_2)$$

(*Mathematically, $\overline{\mathcal{M}_g} \supset \mathcal{M}_{g_1,1}\times\mathcal{M}_{g_2,1}$; note that we in fact get a formula on $\mathcal{M}_{g_1}\times\mathcal{M}_{g_2}$, i.e. on the Satake compactification*)
(*Degeneration to $\mathcal{M}_{g-1,2}\subset\overline{\mathcal{M}_g}$ is not clear — off-shell amplitudes*)

Superstring measure ansatz

$\Xi^{(1)}[m] = \theta_m^4(\tau) \prod \theta_n^4(\tau)$

`D'Hoker-Phong`:

— An explicit expression for $\Xi^{(2)}[m]$ in terms of theta

— Conjectures for $\Xi^{(3)}[m] = \theta_m^4(\tau) \cdot \ldots$.

(*Natural to expect the θ_m^4 for 0-3 point vanishing*)

Theorem (`Cacciatori-Dalla Piazza-van Geemen`)

There exists a unique modular form $\Xi^{(3)}[m]$ satisfying factorization; it is given explicitly, and is not *divisible by $\theta_m^4(\tau)$.*

Theorem (`G.`)

The following modular form of weight 8 satisfies factorization:

$$\Xi^{(g)}[m](\tau) := \sum_{i=0}^{g} (-1)^i 2^{i(i-1)/2} \sum_{V \subset (\mathbb{Z}/2)^{2g};\ \dim V = i} \prod_{n \in V} \theta_{n+m}^{2^{4-i}}$$

(Provided the roots can be chosen consistently)

First properties of the ansatz

For $g \leq 4$ there are no roots involved, so $\Xi^{(g)}[m]$ is well-defined.

The ansatz reproduces the formulas of D'Hoker-Phong for $g = 2$ and Cacciatori-Dalla Piazza-van Geemen for $g = 3$.

Theorem (`Salvati Manni`)

The square roots can be chosen consistently for the above ansatz for $g = 5$, i.e. $\Xi^{(5)}[m]$ is well-defined on $\mathcal{J}_5$.

Theorem (`Dalla Piazza-van Geemen`)

$\Xi^{(4)}[m]$ *is the unique modular form satisfying factorization.*

Theorem (`Oura-Poor-Salvati Manni-Yuen`)

$\Xi^{(5)}[m]$ *extends holomorphically to all of $\mathcal{A}_5$.*

Further physical properties of the superstring measure

Expect the vanishing of the 0-3 point functions under the `Gliozzi-Scherk-Olive` projection, i.e. summing over all even m. The vanishing of the 0-point function (cosmological constant) is

$$\Xi^{(g)}(\tau) := \sum_m \Xi^{(g)}[m](\tau) \equiv 0$$

$\Xi^{(g)}(\tau)$ is a modular form for the entire group $\mathrm{Sp}(2g, \mathbb{Z})$, of weight 8, vanishing on the boundary. It follows by general principles (*slope of the effective divisors on* $\mathcal{M}_g$) that it vanishes on $\mathcal{J}_g$ for $g \leq 4$.

Theorem (G.-Salvati Manni)

For $g \leq 5$ the cosmological constant $\Xi^{(g)}(\tau)$ is proportional to the Schottky-Igusa form

$$F_g(\tau) := 2^g \sum_{m \in (\mathbb{Z}/2)^{2g}} \theta_m^{16}(\tau) - \left(\sum_{m \in (\mathbb{Z}/2)^{2g}} \theta_m^8(\tau) \right)^2$$

Lattice theta functions and physics

In terms of lattice theta functions,

$$\sum \theta_m^{16}(\tau) = \theta_{D_{16}^+}(\tau), \qquad \sum \theta_m^8(\tau) = \theta_{E_8}(\tau),$$

so

$$F_g(\tau) \ \sim \ \theta_{SO(32)}(\tau) - \theta_{E_8 \times E_8}(\tau)$$

Physics conjecture (`Belavin-Knizhnik, D'Hoker-Phong`)

From the duality of the $SO(32)$ and $E_8 \times E_8$ superstring theories one expects F_g to vanish identically on $\mathcal{J}_g$ for any g.

Theorem (Igusa)

$F_g(\tau)$ vanishes identically on $\mathcal{A}_g$ for $g \leq 3$, and gives the defining equation for $\mathcal{M}_4 \subset \mathcal{A}_4$.

Theorem (`G.-Salvati Manni`)

This conjecture is false for any $g \geq 5$.
(in fact the zero locus of F_5 on $\mathcal{M}_5$ is the divisor of trigonal curves)

Higher genus cosmological constant

Thus $\Xi^{(5)}(\tau) = \mathrm{const} \cdot F_5(\tau) \not\equiv 0$ on $\mathcal{J}_5$, i.e. the cosmological constant for the proposed ansatz does not vanish for genus 5.

Observation: G.-Salvati Manni

The modified ansatz

$$\Xi'^{(5)}[m](\tau) := \Xi^{(5)}[m](\tau) - \mathrm{const} F_5(\tau)$$

still satisfies the factorization constraint (F_5 factorizes to identically zero), and gives vanishing cosmological constant

What happens for higher genera?

Further directions and open questions

- Find a holomorphic adjustment of $\Xi^{(6)}[m](\tau)$ satisfying the factorization constraint (to $\Xi'^{(5)}[m](\tau)$), and giving a vanishing cosmological constant. Here subtracting a multiple of F_6 would not work, as it does not factorize to zero.
- Verify the vanishing of two-point function for the proposed ansatz after the GSO projection: known for $g = 2$ (D'Hoker-Phong) and for $g = 3$ (G.-Salvati Manni), trouble for $g = 4$ (Matone-Volpato)
- Verify the vanishing of the three-point function for the proposed ansatz after the GSO projection: known for $g = 2$ (D'Hoker-Phong), trouble for $g = 3$ (Matone-Volpato)
- Compute the (non-vanishing) 4-point function: only known for $g = 2$ (D'Hoker-Phong)
- Apply these modular forms to approach the Schottky problem in genus 5 and higher.