MAT 326 Final — Fall 2006

INSTRUCTOR: SAMUEL GRUSHEVSKY

This is a 24-hour final take-home exam. You are expected to work on it individually. The only materials you are allowed to use during the exam are your own lecture notes from the course, your homeworks, and the textbooks for the course, i.e. Madsen and Tornehave: From Calculus to Cohomology, and the first two chapters of Wells: Differential Analysis on Complex Manifolds. In solving the exam problems you may use the material from the textbooks without proof — then please quote the appropriate statement and give the exact reference.

No other materials (eg. other textbooks or the results of web searches) are allowed.

Please remember to write out and sign the honor pledge on your solutions: *"I pledge my honor that I have not violated the Honor Code during this examination. Signature."*

There are 6 problems in this exam, arranged in what I believe to be roughly the order of increasing difficulty (though you may disagree) — however, parts a) and b) of 6 should not be as hard as c) and d). Problems 1-5 are 15 points each, and problem 6 is 25 points, for the total of 100 points. Partial credit will of course be given for partial solutions. **You do not need to completely solve all the problems to get an "A" on the final.**

Good luck!
Sam

MAT 326 Final — Fall 2006

Problem 1. (10.13 in Madsen and Tornehave) [15 pts]
Let ω be a (smooth) 1-form on a smooth manifold X. Prove that if $\int_\gamma \omega = 0$ for any closed curve γ, then ω is exact (i.e. $\omega = df$ for some function f).

Problem 2. [15 pts]
Show that for each n there exists a trivial vector bundle E of some rank on the sphere S^n, such that the direct sum of bundles (each fiber is the direct sum of vector spaces) $TS^n \oplus E$ is a trivial bundle on S^n (where TS^n is the tangent bundle to S^n).

Problem 3. [15 pts]
Let $f : X \to Y$ be a continuous map of topological spaces. Let $\mathcal{A}$ be a sheaf (of abelian groups) on X; we then define its direct image $f\mathcal{A}$ to be the presheaf on Y given by

$$f\mathcal{A}(U) := \mathcal{A}(f^{-1}(U)).$$

a) Prove that $f\mathcal{A}$ is a sheaf on Y.

Let now $\mathcal{B}$ be a sheaf of abelian groups on Y, which we think of as a sheaf of sections of $\pi : B \to Y$. We define its inverse image by

$$f^*\mathcal{B} = \text{sections of } \{(x,b) \in X \times B | f(x) = \pi(b)\}.$$

b) Prove that $f^*\mathcal{B}$ is a sheaf on X.

c) Construct a natural isomorphism of the spaces of sheaf homomorphisms $Hom(f^*\mathcal{B}, \mathcal{A}) \cong Hom(\mathcal{B}, f\mathcal{A})$.

Problem 4. [15 pts]
The Klein bottle is the smooth manifold obtained by identifying the sides of the square $[0, 2\pi] \times [0, 2\pi]$ in the (x, y)-plane as follows: $(x, 0) \sim (x, 2\pi)$; $(0, y) \sim (2\pi, 2\pi - y)$.

a) Write down an explicit Morse function on the Klein bottle in terms of x and y.

b) Use this Morse function to compute all the cohomology groups of the Klein bottle.

Problem 5. [15 pts]
For which n does the complex projective space $\mathbb{CP}^n$ admit a smooth map to itself without a fixed point?

Problem 6. [25 pts]
Let G denote the set of all two-dimensional vector subspaces in $\mathbb{R}^4$ (i.e. all the 2-planes through the origin).

a) Show that G is a smooth manifold, and compute its dimension.

b) Construct explicitly an embedding of G into $\mathbb{R}^n$, for some n. (Hint: first construct an embedding into some $\mathbb{RP}^m$)

c) Compute the dimensions of the cohomology groups $H^*(G)$.

d) Compute the ring structure on the cohomology, i.e. describe the wedge product maps on $H^*(G)$.