MAT 319 Practice Final Exam.
December 6, 2012

This is a closed notes/ closed book/ electronics off exam.

Each problem is worth 20 points (but the problems are of variable difficulty!).

Problem 1. Prove, using only properties of real numbers, that there exists a real number x such that $x^2 + x = 3$.

Problem 2. Suppose $\{s_n\}$ is a sequence such that for all n, $|s_n - s_{n+1}| < 1/n$. Can the sequence $\{s_n\}$ be convergent? can it be divergent?

Problem 3. Suppose every number of the form 2^n for $n \in \mathbb{Z}$ is a subsequential limit of $\{s_n\}$. Prove that there exists a subsequence of $\{s_n\}$ converging to zero.

Problem 4. Let the function $f : \mathbb{R} \to \mathbb{R}$ be defined to be zero for any irrational number, and to be $f(p/q) := 1/q$ for any rational number p/q with p and q coprime. Is f continuous at 0? at 1?

Problem 5. Let $f : [-1, 1] \to \mathbb{R}$ be defined by $f(x) := \begin{cases} 0 & \text{if } x < 0 \\ x + 1 & \text{if} x \geq 0 \end{cases}$.
Is f continuous on $[-1, 1]$? differentiable on $(-1, 1)$? integrable on $[-1, 1]$? Does it satisfy the conclusions of the Mean Value Theorem and the Intermediate Value Theorem on $[-1, 1]$?

Problem 6. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function, and let $g : \mathbb{R} \to \mathbb{R}$ be some function such that $\lim\limits_{x \to a} (f \circ g)(x) = (f \circ g)(a)$. Does it necessarily follow that $\lim\limits_{x \to a} g(x) = g(a)$?

Problem 7. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a function differentiable everywhere. For any $x \in \mathbb{R}$, compute the limit $\lim\limits_{h \to 0} \frac{f(x+h^2) - f(x)}{h}$.

Problem 8. Find all integrable functions $f : [0, 1] \to \mathbb{R}$ such that for any $x \in [0, 1]$ we have

$$\left(\int_0^x f(t) dt \right)^2 = \int_0^x \left(f(t)^3 \right) dt.$$