MAT 319 Practice Final Exam.
December 6, 2012

This is a closed notes/ closed book/ electronics off exam.
Each problem is worth 20 points (but the problems are of variable difficulty!).

**Problem 1.** Prove, using only properties of real numbers, that there exists a real number $x$ such that $x^2 + x = 3$.

**Problem 2.** Suppose $\{s_n\}$ is a sequence such that for all $n$, $|s_n - s_{n+1}| < 1/n$. Can the sequence $\{s_n\}$ be convergent? can it be divergent?

**Problem 3.** Suppose every number of the form $2^n$ for $n \in \mathbb{Z}$ is a subsequential limit of $\{s_n\}$. Prove that there exists a subsequence of $\{s_n\}$ converging to zero.

**Problem 4.** Let the function $f : \mathbb{R} \rightarrow \mathbb{R}$ be defined to be zero for any irrational number, and to be $f(p/q) := 1/q$ for any rational number $p/q$ with $p$ and $q$ coprime. Is $f$ continuous at 0? at 1?

**Problem 5.** Let $f : [-1, 1] \rightarrow \mathbb{R}$ be defined by $f(x) := \begin{cases} 0 & \text{if } x < 0 \\ x + 1 & \text{if } x \geq 0 \end{cases}$. Is $f$ continuous on $[-1, 1]$? differentiable on $(-1, 1)$? integrable on $[-1, 1]$? Does it satisfy the conclusions of the Mean Value Theorem and the Intermediate Value Theorem on $[-1, 1]$?

**Problem 6.** Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function, and let $g : \mathbb{R} \rightarrow \mathbb{R}$ be some function such that $\lim_{x \to a} (f \circ g)(x) = (f \circ g)(a)$. Does it necessarily follow that $\lim_{x \to a} g(x) = g(a)$?

**Problem 7.** Suppose $f : \mathbb{R} \rightarrow \mathbb{R}$ is a function differentiable everywhere. For any $x \in \mathbb{R}$, compute the limit $\lim_{h \to 0} \frac{f(x+h^2)-f(x)}{h}$.

**Problem 8.** Find all integrable functions $f : [0, 1] \rightarrow \mathbb{R}$ such that for any $x \in [0, 1]$ we have

$$\left( \int_0^x f(t) dt \right)^2 = \int_0^x \left( f(t)^3 \right) dt.$$