Riemann surfaces

Junior seminar, Princeton University, Spring 2005

Samuel Grushevsky

A Riemann surface can be thought of as a complex manifold of (complex) dimension one, as a real manifold of dimension two endowed with a metric of constant curvature, as an algebraic curve over $\mathbb{C}$, or in terms of function field extensions. Riemann surfaces, as the object of study, or as a tool to approach other questions, appear in many branches of mathematics.

The goal of this tutorial is to introduce the Riemann surfaces in the context of complex analysis and algebraic geometry. We will first study the topological properties of Riemann surfaces and then proceed to discuss the geometric objects associated to a Riemann surface — those objects that keep track of the complex/metric structure on a surface and not only of the underlying real manifold.

Knowledge of basic complex analysis (holomorphic functions, maximum principle, …) would be helpful for this course.

Please contact me at `sam@math.princeton.edu` with any questions.

Syllabus

- Definition of a Riemann surface as a complex manifold.
- Methods of defining a Riemann surface, and examples of Riemann surfaces.
- Holomorphic and meromorphic functions on Riemann surfaces.
- Topological classification of Riemann surfaces; genus.
- Branched covers of Riemann surfaces; Riemann-Hurwitz formula.
- Meromorphic forms on Riemann surfaces; residues.

Suggested student projects

- Uniformization theory and geometry of hyperbolic Riemann surfaces.
- Divisors on a Riemann surface; Riemann-Roch formula and Serre's duality for Riemann surfaces.
- Period matrix and the Jacobian of a Riemann surface; Abel-Jacobi map and Abel's theorem.
- Theta functions and their addition theorem. Constructing functions on Riemann surfaces with prescribed zeroes.
- Weierstrass points on Riemann surfaces; hyperelliptic curves and general curves with automorphisms.
- Families of Riemann surfaces / algebraic curves and their moduli.