Strong Curvature Positivity of Holomorphic Hilbert Bundles

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Setting

- Consider a bounded domain $\Omega \subset \mathbb{C}^n$ and a domain $U \subset \mathbb{C}^m$.
- Let φ be a weight function on $\Omega \times U$ which is smooth up to the boundary, define $\varphi_t(\cdot) := \varphi(\cdot, t)$, and let

$$L^2_{\varphi_t}(\Omega) := \left\{ f : \Omega \to \mathbb{C}; \int_\Omega |f|^2 e^{-\varphi_t} dV_\Omega < +\infty \right\} \text{ and } \mathcal{H}^2_{\varphi_t}(\Omega) := L^2_{\varphi_t}(\Omega) \cap \mathcal{O}(\Omega).$$

- Since Ω and U are bounded and φ is smooth up to the boundary on $\Omega \times U$, the Hilbert spaces $\mathcal{H}^2_{\varphi_t}(\Omega)$ are independent of t as subspaces of $\mathcal{O}(\Omega)$.
- Thus one has a vector bundle E over U whose fiber at $t \in U$ is $\mathcal{H}^2_{\varphi_t}(\Omega)$. This vector bundle is trivial as a holomorphic vector bundle, but has a non-trivial Hermitian metric given by the $L^2_{\varphi_t}$-norm on $\mathcal{H}^2_{\varphi_t}(\Omega)$ in the fiber over t.
- The goal is to find conditions under which the curvature of E is positive.

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Berndtsson's Nakano-positivity result

Theorem (Berndtsson)

If Ω is pseudoconvex and φ is (strictly) plurisubharmonic on $\Omega \times U$, then the holomorphic hermitian bundle $\left(E, \|\cdot\|_{\varphi_t}\right)$ is (strictly) Nakano-positive.

- Pseudoconvex: $\mathbb{C}$-version of a convex domain.
- Plurisubharmonic: $\mathbb{C}$-version of a convex function, meaning $\left[\frac{\partial^2 \varphi}{\partial z_i \partial z_j}\right]_{i,j}$ is positive definite.

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A brief discussion of positivity

- In complex geometry, positive curvature is a notion analogous to convexity.
- Strongest version of such positivity is called Nakano-positivity. Weakest version is Griffiths positivity.
- If the base *U* has dimension 1, both versions coincide.
- *E* is Griffiths-positive $\iff \log\left(\|\xi\|^2\right)$ is plurisubharmonic for every holomorphic section ξ of E^*.

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Applications of Berndtsson's theorem

Berdtsson's theorem has two important applications:

1. Convexity of the Mabuchi K-energy along geodesics in the space of Kähler metrics. (Berndtsson.)
2. Crucial tool in the proof of an optimal L^2 extension theorem for holomorphic functions. (Berndtsson-Lempert.)
    - L^2 extension is a fundamental result in complex analysis and geometry.
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Main result

Theorem (A.)

If Ω is pseudoconvex and has a function η such that $-e^{-\eta}$ is plurisubharmonic, then $\left(E, \|\cdot\|_{\varphi_t}\right)$ is Nakano-positive, provided that the $\mathbb{C}$-Hessian of φ has a precise negative lower bound determined by η.

- In particular, if Ω has a negative plurisubharmonic function η, then φ does not have to be plurisubharmonic.
- Remark: Every bounded pseudoconvex domain has at least one such function η.

The main theorem holds (and was proved) in the setting of Stein manifolds.

- Ω a bounded pseudoconvex in a Stein manifold.
- Holomorphic functions replaced by holomorphic sections of a line bundle.
- Weights replaced by metrics for line bundle.
- The assumptions on the Hessian of φ replaced by corresponding curvature assumptions on the metrics for the line bundle.

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Example: the unit ball in $\mathbb{C}^n$

The function given by $\eta(z) = -\log(1 - |z|^2)$ on the unit ball $\mathbb{B}_n(1) \subset \mathbb{C}^n$ is negative and plurisubharmonic. Its Hessian is (up to a constant) ω_B, the Bergman metric for $\mathbb{B}_n(1)$.

Example

Let φ be defined on $\mathbb{B}_n(1) \times \mathbb{D}$ by $\varphi(z, t) = \frac{1}{2}\log(1 - |z|^2) + |t|^2$. Then:

$$\text{Hess}_{\mathbb{C}}(\varphi) = -\frac{1}{2}\omega_B(z) + \omega_E(t)$$

is not positive-definite but still satisfies the hypotheses of our theorem.

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Going further

- Have proved a theorem on the variation of Bergman kernels (important objects in the theory of several complex variables).
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Thank you for your attention.