Spring 2017 MAT 536, Complex Analysis
Instructor: Samuel Grushevsky
Homework #5, due in class Wed March 1

Problem 1. (Recall that) a function $f : \mathbb{CP}^1 \to \mathbb{C}$ is said to be holomorphic at ∞, if the function $f(1/z)$ is holomorphic at $z = 0$.

If $P(z)$ and $Q(z)$ are two polynomials and $\deg P \le \deg Q$, prove that the ratio $P(z)/Q(z)$ is holomorphic at ∞.

Problem 2. Prove that if a function f is *meromorphic* on all of $\mathbb{CP}^1$, then f is a rational function — that is, f is equal to the ratio of some two polynomials.

Problem 3. (Recall that Δ denotes the open unit disk)

Prove that the group of biholomorphic maps (holomorphic bijections, such that the inverse is also holomorphic) $\Delta \to \Delta$ is isomorphic to $\mathrm{PSL}_2(\mathbb{R})$. What is the group of biholomorphic maps $\mathbb{C} \to \mathbb{C}$?

Problem 4. (Recall that $\mathbb{H}$ denotes the upper half-plane)

Suppose that $f : \mathbb{H} \to \Delta$ is a holomorphic function such that $f(ni) = 0$ for any $n \in \mathbb{Z}_{>0}$. Prove that f is identically zero.

Problem 5. For each of the following functions, determine whether they have a removable singularity, a pole, or an essential singularity at $z = \infty$. If it is a removable singularity, what is the value at infinity (and if it's 0, what is the order of the zero?). If it is a pole, what is the order of the pole?

$$\frac{z^3+1}{z^5+2};$$

$$\sinh z := \frac{e^z - e^{-z}}{2};$$

$$\frac{e^z}{z^3};$$

$$e^{\frac{z-1}{z}} - e.$$