Spring 2017 MAT 536, Complex Analysis
Instructor: Samuel Grushevsky
Homework #3, due in class Wed February 15

Problem 1. Prove rigorously that the action of $PSL_2(\mathbb{C})$ on $\mathbb{CP}^1$ is three-transitive, i.e. that for any two triples of distinct points $z_1 \neq z_2 \neq z_3 \neq z_1$ and $w_1 \neq w_2 \neq w_3 \neq w_1$ in $\mathbb{CP}^1$, there exists a $\gamma \in PSL_2(\mathbb{C})$ such that $\gamma(z_i) = w_i$ for $i = 1, 2, 3$. *Hint: first do* $w_1 = 0, w_2 = \infty, w_3 = 1$,

Problem 2. Prove that given two ordered quadruples (z_1, z_2, z_3, z_4) and (w_1, w_2, w_3, w_4) of distinct points of $\mathbb{CP}^1$, there exists a $\gamma \in PSL_2(\mathbb{C})$ such that $\gamma(z_i) = w_i$ for $i = 1, 2, 3, 4$ if and only if the cross-ratios of the quadruples are equal. *Hint: compute that the cross-ratio is invariant under the action of* $PSL_2(\mathbb{C})$ *and use problem 1.*

Problem 3. Give a three-line proof that the action of $PSL_2(\mathbb{R})$ on the upper half-plane $\mathbb{H}$ is not three-transitive.

Problem 4. Write down the Taylor series expansion of $\ln(1 + x)$ for $x \in \mathbb{R}$, and use this power series to define the logarithm of a complex number. What is the radius of convergence of this power series, and what is the image of the map $\ln(1 + z)$ from the interior of its disk of convergence to $\mathbb{C}$?

Problem 5. Prove that if a function f is holomorphic on all of C, and is not a constant, then the image $f(\mathbb{C})$ is dense in $\mathbb{C}$.