MAT 211: INTRODUCTION TO LINEAR ALGEBRA

Answer Keys to the Practice Midterm 1

If you find any mistake in the following answer keys, please do let me know via email. The instructor is not responsible of any possible mistake in these notes.

Problem 1: a) $x = 3, \quad y = 23, \quad z = -5.$

b)

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = t \begin{pmatrix} 4/3 \\ 1 \\ 0 \end{pmatrix} + s \begin{pmatrix} -2/3 \\ 0 \\ 1 \end{pmatrix}$$

where t and s are free variables. You can also say that the space of solutions is the span of $\begin{pmatrix} 4/3 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -2/3 \\ 0 \\ 1 \end{pmatrix}$, which is a plane in $\mathbf{R}^3$.

Problem 2: a) If $k = 1$ the solutions are

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = t \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + s \begin{pmatrix} -3/5 \\ -1/5 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} -3/5 \\ 4/5 \\ 0 \\ 0 \end{pmatrix}$$

where t and s are free variables. In this case there are ∞^2-many solutions. The solutions form a plane in $\mathbf{R}^4$ (not passing thorough the origin).

If $k \neq 1$, the solutions are

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = t \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} -3/5 \\ 4/5 \\ 0 \\ 0 \end{pmatrix}$$

where t is a free variable. In this case there are ∞^1-many solutions. The solutions form a line, not passing through the origin, in $\mathbf{R}^4$.

b) For any value of k there is only one solution $x = y = z = k/2$.

Problem 3:The matrix A is not invertible only when either $a = 0$ or $a = 3$. If $a = 1$, the inverse of A is

$$\begin{pmatrix} 1 & -2 & -1/2 \\ 0 & 1 & 1 \\ 0 & 0 & -1/2 \end{pmatrix}$$

Problem 4: a) $\begin{pmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{pmatrix}$

b) $\begin{pmatrix} -3/5 & 4/5 \\ 4/5 & 3/5 \end{pmatrix}$

c) $\begin{pmatrix} -3/5 & 4/5 \\ 4/5 & 3/5 \end{pmatrix} \begin{pmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{pmatrix} = \frac{1}{10} \begin{pmatrix} -3\sqrt{3}+4 & 3+4\sqrt{3} \\ 4\sqrt{3}+3 & -4+3\sqrt{3} \end{pmatrix}.$

d) $\frac{1}{10} \begin{pmatrix} \sqrt{3}-7 \\ 7\sqrt{3}+1 \end{pmatrix}$

Problem 5:

$A = \begin{pmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{pmatrix} \begin{pmatrix} 4/5 & 2/5 \\ 2/5 & 1/5 \end{pmatrix} = \frac{1}{5} \begin{pmatrix} \sqrt{2} & \sqrt{2}/2 \\ 3\sqrt{2} & 3\sqrt{2}/2 \end{pmatrix}$. The matrix A is not invertible. $\text{Ker}(T) = \text{span}\{\begin{pmatrix} -1 \\ 2 \end{pmatrix}\}$. $\text{Im}(T) = \text{span}\{\begin{pmatrix} 1 \\ 3 \end{pmatrix}\}$.

Problem 6:

a)

$$A = \frac{1}{9}\begin{pmatrix} 4 & 2 & 4 \\ 2 & 1 & 2 \\ 4 & 2 & 4 \end{pmatrix}.$$

b) $\vec{r} = \frac{2}{3}\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$.

c) All vectors of the form

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = t\begin{pmatrix} -1/2 \\ 1 \\ 0 \end{pmatrix} + s\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

where t and s are free variables are perpendicular to both $\vec{r}$ and $\vec{w}$. Therefore the vectors perpendicular to both $\vec{r}$ and $\vec{w}$ form a plane in $\mathbf{R}^3$.

Problem 7:

a)

$$\frac{1}{a^2+b^2+c^2}\begin{pmatrix} b^2+c^2 & -ab & -ac \\ -ab & a^2+c^2 & -bc \\ -ac & -bc & a^2+b^2 \end{pmatrix}.$$

b) $\frac{1}{3}\begin{pmatrix} 7 \\ 2 \\ 5 \end{pmatrix}$

Problem 8: a)

$$A = \begin{pmatrix} 1 & 1 & -1 \\ 0 & -1 & 1 \\ -2 & -2 & 2 \end{pmatrix}.$$

b) $\mathrm{Im}(T) = \mathrm{span}\{\begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}\}$.

c) $\mathrm{Ker}(T) = \mathrm{span}\{\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\}$.

Problem 9: a) $\mathrm{Im}(A) = \mathrm{span}\{\begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix}, \begin{pmatrix} -2 \\ -5 \\ -2 \end{pmatrix}\}$.

b) $\mathrm{Ker}(A) = \mathrm{span}\{\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}\}$.

Problem 10: a) $\mathrm{Im}(A) = \mathrm{span}\{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix}\}$.

b) $\mathrm{Ker}(A) = \mathrm{span}\{\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 0 \\ 1 \end{pmatrix}\}$.

Problem 11: a)

$$R_z(180)R_x(90)\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}.$$

b)

$$R_z(45)R_y(30)\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} \sqrt{6}-\sqrt{2} \\ \sqrt{6}+3\sqrt{2} \\ 2\sqrt{3}-2 \end{pmatrix}.$$