Practice Midterm 1

Problem 1. Solve the following systems using augmented matrices. State whether the solution is unique, there are no solutions, or whether there are infinitely many. If the solution is unique give it. If there infinitely many give the solution parametrically, namely in terms of the free variables.

$$\begin{cases} x_1 \qquad\quad - x_3 = 8 \\ 2x_1 + 2x_2 + 9x_3 = 7 \\ \qquad\quad x_2 + 5x_3 = -2 \end{cases}$$

$$\begin{cases} 3x_1 - \ \ 4x_2 + 2x_3 = 0 \\ -9x_1 + 12x_2 - 6x_3 = 0 \\ -6x_1 + \ \ 8x_2 - 4x_3 = 0 \end{cases}$$

Problem 2. Discuss the number of solutions of the following systems depending on the real parameter k. Moreover when the solution is unique, or there are infinitely many solutions, write all the solutions in parametric form.

$$\begin{bmatrix} x_1 & + & 2x_2 & - & x_3 & + & kx_4 & = & 1 \\ -2x_1 & + & x_2 & + & 2x_3 & - & x_4 & = & 2 \\ 4x_1 & + & 3x_2 & - & 4x_3 & + & 3x_4 & = & 0 \end{bmatrix}$$

$$\begin{bmatrix} & & y & + & z & = & k \\ x & & & + & z & = & k \\ x & + & y & & & = & k \end{bmatrix}$$

Problem 3. Say for which values of the real parameter a the following matrix is invertible:

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & a & 2 \\ 0 & 0 & a^2 - 3a \end{pmatrix}.$$

Then set $a = 1$ and find the inverse.

Problem 4. a) Write the matrix representing a linear transformation that rotates vectors of $\mathbf{R}^2$ by 30 degrees counterclockwise.

b) Write the matrix representing a linear transformation that reflects vectors of $\mathbf{R}^2$ about the line $y = 2x$.

c) Write the matrix representing a linear transformation that first rotates vectors by 30 degrees counterclockwise, and then reflects them about the line $y = 2x$.

d) Find the vector obtained by first reflecting $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ about the line $y = 2x$, and then rotating it by 30 degrees counterclockwise.

Problem 5. Let $T\colon \mathbf{R}^2 \to \mathbf{R}^2$ be the orthogonal projection onto the line $x - 2y = 0$ followed by a counterclockwise rotation by 45 degrees. Find the matrix A that represents T. Is A invertible? Show on a picture the kernel and the image of T.

Problem 6. (Orthogonal projection in $\mathbf{R}^3$.) Recall that the orthogonal projection of a vector $\vec{x}$ in $\mathbf{R}^3$ onto a line L of $\mathbf{R}^3$ is defined as $\text{proj}_L(\vec{x}) = (\vec{x} \cdot \vec{u})\,\vec{u}$, where $\vec{u}$ is a unit vector parallel to L. Alternatively, if instead of a unit vector $\vec{u}$ we have an arbitrary non-zero vector $\vec{w}$ parallel to L, the projection of $\vec{x}$ onto L is defined as

$$\text{proj}_L(\vec{x}) = \left(\frac{\vec{x} \cdot \vec{w}}{\vec{w} \cdot \vec{w}}\right)\vec{w}.$$

Let now $T\colon \mathbf{R}^3 \to \mathbf{R}^3$ be the orthogonal projection onto the line L spanned by the vector $\vec{w} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$.

a) Write the matrix A that represents T.

b) Find the orthogonal projection $\vec{r}$ of the vector $\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ onto L.

c) Find all vectors in $\mathbf{R}^3$ that are perpendicular to $\vec{w}$ and $\vec{r}$. Write them in parametric form (namely in terms of free variables).

Problem 7. (Orthogonal Projections onto a plane of $\mathbf{R}^3$.) The orthogonal projection $\text{proj}_V(\vec{x})$ of a vector $\vec{x}$ in $\mathbf{R}^3$ onto a plane V in $\mathbf{R}^3$ of equation $ax_1 + bx_2 + cx_3 = 0$ is given by the formula:

$$\text{proj}_V(\vec{x}) \;=\; \vec{x} - \left(\frac{\vec{x}\cdot\vec{r}}{\vec{r}\cdot\vec{r}}\right)\vec{r}, \qquad \text{where } \vec{r} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

Note that the 'dot' in the previous formula denotes the dot product of vectors in $\mathbf{R}^3$.
a) Write the matrix that represents the linear transformation proj_V.
b) Find the orthogonal projection of $\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$ onto the plane $x_2 - x_1 + x_3 = 0$ in $\mathbf{R}^3$.

Problem 8. Consider the following linear transformation $T\colon \mathbf{R}^3 \to \mathbf{R}^3$ given by

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} \;=\; \begin{pmatrix} x+y-z \\ -y+z \\ -2x-2y+2z \end{pmatrix}.$$

a) Find the matrix A that represents T.
b) Write the kernel of T as a span of a minimal set of generators.
c) Write the image of T as a span of a minimal set of generators.

Problem 9. Consider the following matrix:

$$A = \begin{pmatrix} 2 & -2 & -4 \\ -3 & -5 & -2 \\ 4 & -2 & -6 \end{pmatrix}.$$

a) Write the image of A as a span of a minimal set of generators.
b) Write the kernel of A as a span of a minimal set of generators.

Problem 10. Consider the following matrix:

$$A = \begin{pmatrix} 1 & 2 & 3 & 3 \\ 1 & 2 & 4 & 3 \\ 1 & 2 & 5 & 3 \end{pmatrix}.$$

a) Write the image of A as a span of a minimal set of generators.
b) Write the kernel of A as a span of a minimal set of generators.

Problem 11. (Rotations in $\mathbf{R}^3$.) Consider $\mathbf{R}^3$ with coordinates (x, y, z). The matrix $R_x(\theta)$ that represents the linear transformation $T_{x,\theta}\colon \mathbf{R}^3 \to \mathbf{R}^3$ rotating vectors in $\mathbf{R}^3$ by θ degrees counterclockwise about the x-axis is:

$$R_x(\theta) \;=\; \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{pmatrix}.$$

Similarly we can define $R_y(\theta)$ and $R_z(\theta)$ which are the matrices that rotate vectors by θ degrees counterclockwise about the y- and z-axis, respectively:

$$R_y(\theta) \;=\; \begin{pmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{pmatrix}, \qquad R_z(\theta) \;=\; \begin{pmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

a) Find the vector obtained by rotating $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ first by 90 degrees counterclockwise about the x-axis, and then by rotating it by 180 degrees counterclockwise about the z-axis.
b) Find the vector obtained by rotating $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ first by 30 degrees counterclockwise about the y-axis, and then by rotating it by 45 degrees counterclockwise about the z-axis.