Practice Problems for Midterm II

Problem 1: Find a basis $\mathcal{B}$ for the following subspace of $\mathbf{R}^4$

$$U = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 5 \\ 2 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} \right\}.$$

Find the dimension of U. Find the $\mathcal{B}$-coordinates of the vector $\vec{w} = \begin{pmatrix} 7 \\ 3 \\ 3 \\ 3 \end{pmatrix}$.

Problem 2: Let $(\vec{u}, \vec{v}, \vec{w})$ be a basis of $\mathbf{R}^3$. Say for which values of the real parameter k the following vectors form a basis of $\mathbf{R}^3$:

$$\vec{u} + \vec{v} + \vec{w}, \qquad \vec{u} - \vec{v} + \vec{w}, \qquad \vec{u} + k\vec{v} + k^2\vec{w}.$$

Problem 3: Find a basis of the subspace U in $\mathbf{R}^4$ defined by the equations $x_1 + 2x_2 - 3x_3 + x_4 = 0$ and $2x_1 - x_3 - 2x_4 = 0$. Find moreover a basis of the orthogonal complement $U^\perp$ of U (in other words find a basis of the subspace of $\mathbf{R}^4$ consisting of all vectors perpendicular to U).

Problem 4:

Let T be the linear operator on R^3 defined by

$$T(x_1, x_2, x_3) = (3x_1 + x_3, -2x_1 + x_2, -x_1 + 2x_2 + 4x_3).$$

(a) What is the matrix of T in the standard ordered basis for R^3?

(b) What is the matrix of T in the ordered basis

$$\{\alpha_1, \alpha_2, \alpha_3\}$$

where $\alpha_1 = (1, 0, 1)$, $\alpha_2 = (-1, 2, 1)$, and $\alpha_3 = (2, 1, 1)$?

Problem 5. **(1)**. Find an orthonormal basis for the plane in $\mathbf{R}^4$ spanned by the vectors $(1,1,1,1)$ and $(1,9,-5,3)$. **(2)**. Find an orthonormal basis of $\mathbf{R}^3$ starting from the vectors $(1,1,1)$, $(1,0,1)$ and $(0,1,-1)$. **(3)**. Find an orthonormal basis for the plane in $\mathbf{R}^3$ defined by $x + y + z = 0$ (find first a basis for the plane).

Problem 6:

Find an orthogonal matrix of the form

$$\begin{bmatrix} 2/3 & 1/\sqrt{2} & a \\ 2/3 & -1/\sqrt{2} & b \\ 1/3 & 0 & c \end{bmatrix}.$$

Problem 7:

Find the orthogonal projection of

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

onto the subspace of $\mathbb{R}^4$ spanned by

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}.$$