MAT 331: Project 2

Project description

In this project we will use Mathematica to solve several problems involving differential equations. Differential equations are currently used to model a wide range of phenomena from biology, physics, chemistry, computer science, economic analysis, etc. The theory of differential equations has become an essential tool in all areas of science, particularly since computers became commonly available.

Problem 1

The fish and game department in a certain state is planning to issue hunting permits to control the deer population (one deer per permit). It is known that if the deer population falls below a certain level m, the deer will become extinct. It is also known that if the deer population rises above the carrying capacity M, the population will decrease back to M through disease and malnutrition. Consider the following model for the growth rate of the deer population P as a function of time t:

\[
\frac{dP}{dt} = rP(M - P)(P - m)
\]

where P is the deer population and r=0.00003 is a constant of proportionality. The values of the other parameters are M=100 and m=55.

1. Plot the vector field of the differential equation using VectorPlot[...]. Then plot the vector field using StreamPlot[...]; identify the constant solutions and color them in red using StreamPlot[...]. Try a window size of 100x130 (in the tP plane). What happens to the solutions as time t increases, t→∞? Color a couple of trajectories to illustrate the possible behaviors, then explain the plot.

2. Solve the system of differential equations for the initial condition P(20)=110. If DSolve[...] does not work, then a numeric approximation may be the next best thing to hope for, so use Mathematica to find a numerical approximation of the true solution, then plot it.

3. If the initial deer population is 140, about how many hunting permits should be issued so that the deer population does not become extinct?

4. Write a small interactive model using Manipulate[...] and Locator[...], that initially plots the StreamPlot[...] from part 1, and then on click, colors the solution curve that passes through the point where the user clicked.

Problem 2

In this problem, we will visualize a famous phenomenon in dynamics of nonlinear differential equations, called the Poincaré-Andronov-Hopf bifurcation. It exhibits the birth of a limit cycle through a change in the stability of the equilibrium point. Consider the nonlinear system S of differential equations given below, where \( \alpha \) is a real parameter:
\[ \begin{align*}
x' &= y - x(x^2 + y^2 - \alpha) \quad (1) \\
y' &= -x - y(x^2 + y^2 - \alpha) \quad (2)
\end{align*} \]  

Its linearization at \((0, 0)\) is denoted by \( \mathcal{L} \) and given by \[ \begin{align*}
x' &= \alpha x + y \quad (1) \\
y' &= -x + \alpha y \quad (2)
\end{align*} \]

1. Find the equilibrium points of the nonlinear system. Find the eigenvalues of the coefficient matrix of the linearized system at the equilibrium points in terms of the parameter \( \alpha \).

2. Look at the linear system \( \mathcal{L} \) and assume that the value of the parameter \( \alpha \) is \(-1/4\).

2.0.1. Assume that the system satisfies the initial condition \( x(0) = -10 \) and \( y(0) = 20 \). Use \texttt{DSolve[...]} to find the solution of the corresponding initial value problem, then plot the corresponding trajectory in the \( xy \) plane.

2.0.2. For your trajectory in part 2.0.1, draw the graphs of \( x \) versus \( t \) and \( y \) versus \( t \) on the same graph.

2.0.3. For your trajectory in part 2.0.1, draw the corresponding graph in the three-dimensional \( txy \)-space.

\( \triangleright \) In parts 2.0.1 - 2.0.3, it is important that you choose appropriate ranges for \( x, y, t \) and appropriate scales for your plot.

3. By using \texttt{Manipulate[...]} and \texttt{StreamPlot[...]}, do an interactive model of the vector field of the system, with the parameter \( \alpha \) taking values in the closed interval \([-2, 2]\). The interactive model will show the Phase Portrait of the nonlinear system \( \mathcal{S} \), the Phase Portrait of the linear system \( \mathcal{L} \) and the eigenvalues at the equilibrium points.

4. Analyze the type and stability of the equilibrium points of the nonlinear system \( \mathcal{S} \) for all values of the parameter \( \alpha \). Use the interactive model to find the values of the parameter \( \alpha \) where the qualitative nature of the solutions for the system changes.

5. Use the Interactive model to find the values of the parameter \( \alpha \) for which the system develops limit cycles. Then prove mathematically the existence of the limit cycle by changing the system into polar coordinates and solving it.

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**General considerations:**

No previous knowledge of differential equations is assumed in this project, except for the theory developed in the lecture notes posted on Blackboard. Please read the lab notes on Blackboard to find what \textit{Mathematica} commands are used for solving differential equations and visualizing Phase Portraits. More examples and tutorials about solving differential equations and plotting vector fields can be found in the \textit{Mathematica} documentation (Help \( \rightarrow \) Wolfram Documentation).