Problem 4: Let \( \{f_n\} \) be a sequence of measurable functions defined on \( \mathbb{R} \). Show that the sets

\[
E_1 = \{ x \in \mathbb{R} \mid \lim_{n \to \infty} f_n(x) \text{ exists and is finite} \}
\]

\[
E_2 = \{ x \in \mathbb{R} \mid \lim_{n \to \infty} f_n(x) = \infty \}
\]

\[
E_3 = \{ x \in \mathbb{R} \mid \lim_{n \to \infty} f_n(x) = -\infty \}
\]

are measurable.

Solution. Theorem 3.5 of the textbook says that if \( \{f_n\} \) is a sequence of measurable functions, then the functions \( g = \liminf_{n \to \infty} f_n \) and \( h = \limsup_{n \to \infty} f_n \) are measurable.

Notice that \( \lim_{n \to \infty} f_n(x) = \infty \), if and only if \( \liminf_{n \to \infty} f_n(x) = \infty \). Hence,

\[
E_2 = \{ x \mid g(x) = \infty \} = \bigcap_{k \in \mathbb{N}} \{ x \mid g(x) > k \}
\]

is measurable. Likewise,

\[
E_3 = \{ x \mid h(x) = -\infty \} = \bigcap_{k \in \mathbb{N}} \{ x \mid h(x) < -k \}
\]

is measurable.

Further, notice that \( E_1 = \{ x \in |g(x) = h(x)\} \setminus (E_2 \cup E_3) \), hence \( E_1 \) is also measurable. \( \square \)

Problem 2: Let \( C \subset [0,1] \) be the Cantor middle-thirds set. Suppose that \( f : [0,1] \to \mathbb{R} \) is defined by \( f(x) = 0 \) for \( x \in C \) and \( f(x) = k \) for all \( x \) in each interval of length \( 3^{-k} \) which has been removed from \([0,1] \) at the \( k^{th} \) step of the construction of the Cantor set. Show that \( f \) is measurable and calculate \( \int_{[0,1]} f \, dm \).

Solution. Denote by \( f_n : [0,1] \to \mathbb{R} \) the function constructed following way: If \( C_k \) denotes the union of the intervals of length \( 3^{-k} \) removed in the \( k \)-th step of the construction of the Cantor middle-third set, let \( f_n(x) = k \) for \( x \in C_k \), and zero elsewhere. Then \( f_n \) is a simple function (it only takes \( n+1 \) values). Furthermore, it is easy to see that \( f_n \to f \) pointwise, hence \( f \) is a measurable function. In addition, the sequence \( f_n \) is increasing to \( f \), hence the Monotone Convergence Theorem gives us

\[
\int_{[0,1]} f \, dm = \lim_{n \to \infty} \int_{[0,1]} f_n \, dm = \lim_{n \to \infty} \left( \sum_{k=1}^{n} k 2^{k-1} 3^{-k} \right) = \frac{1}{3} \lim_{n \to \infty} \left[ \sum_{k=1}^{n} k \left( \frac{2}{3} \right)^{k-1} \right]
\]

The answer up to this point is fine. With a little more effort, one can get the answer 3. This uses the following relation:

\[
\sum_{k=1}^{\infty} k x^{k-1} = \frac{1}{(1-x)^2} \quad \text{if} \quad 0 < |x| < 1
\]
There are a number of ways one can use to prove this fact, including Riemman sums and Taylor’s formula.

**Problem 3:** Let $E$ be a measurable set. For a function $f : E \rightarrow \mathbb{R}$ we define the positive part $f^+ : E \rightarrow \mathbb{R}$, $f^+(x) = \max(f(x), 0)$, and the negative part $f^- : E \rightarrow \mathbb{R}$, $f^-(x) = \min(f(x), 0)$. Prove that $f$ is measurable if and only if both $f^+$ and $f^-$ are measurable.

**Proof.** One could directly apply the definition of a measurable function or use Theorem 3.5 for the maximum/minimum of two functions $f(x)$ and $g(x) = 0$.

**Problem 4:** Prove that if $f$ is integrable on $\mathbb{R}$ and $\int_E f(x) \, dm \geq 0$ for every measurable set $E$, then $f(x) \geq 0$ a.e. $x$.

**Solution.** Since $f$ is integrable, it is in particular measurable. Let $E$ be the measurable set $E = \{x| f(x) < 0\}$. By hypothesis, and using monotonicity of the integral

$$0 \leq \int_E f(x) \, dm \leq \int_E 0 \, dm = 0 \Rightarrow \int_E f(x) \, dm = 0$$

Notice that $-f$ is a positive function on $E$, and

$$\int_E (-f(x)) \, dm = 0.$$ 

Now Theorem 4.4 implies that $-f$ is zero almost everywhere. By the definition of $E$, this happens if and only if $E$ has zero measure.

**Problem 5:** Let $E$ be a measurable set. Suppose $f \geq 0$ and let $E_k = \{x \in E \mid 2^k < f(x) \leq 2^{k+1}\}$ for any integer $k$. If $f$ is finite almost everywhere, then

$$\bigcup_{k=-\infty}^{\infty} E_k = \{x \in E \mid f(x) > 0\},$$

and the sets $E_k$ are disjoint.

(a) Prove that $f$ is integrable if and only if $\sum_{k=-\infty}^{\infty} 2^k m(E_k) < \infty$.

(b) Let $a > 0$ and consider the function

$$f(x) = \begin{cases} \frac{|x|^a}{a} & \text{if } |x| \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

Use part a) to show that $f$ is integrable on $\mathbb{R}$ if and only if $a < 1$.

**Solution.**

(a) Suppose $f$ is integrable. Since $f(x) > 2^k$ on $E_k$, we have

$$\int_{E_k} f \, dm \geq \int_{E_k} 2^k \, dm = 2^k m(E_k)$$

(b) For $a < 1$, the function $f(x)$ is integrable because $|x|^a$ is integrable on $(-1, 1)$ and $f(x) = 0$ elsewhere. Conversely, if $f(x)$ is integrable, then $|x|^a$ is also integrable on $(-1, 1)$, and $f(x) = 0$ elsewhere, so $a < 1$.
Therefore, by the comparison test,

\[ \sum_{k=-\infty}^{\infty} 2^k m(E_k) \leq \sum_{k=-\infty}^{\infty} \int_{E_k} f dm = \int_{\mathbb{R}} f dm < \infty \]

Next suppose \( \sum_{k=-\infty}^{\infty} 2^k m(E_k) < \infty \). Then \( 2 \left( \sum_{k=-\infty}^{\infty} 2^k m(E_k) \right) = \sum_{k=-\infty}^{\infty} 2^{k+1} m(E_k) < \infty \).

Since \( f(x) \leq 2^{k+1} \) on \( E_k \), we have

\[ \int_{E_k} f dm \leq \int_{E_k} 2^{k+1} dm = 2^{k+1} m(E_k). \]

Then

\[ \int_{\mathbb{R}} f dm = \sum_{k=-\infty}^{\infty} \int_{E_k} f dm \leq \sum_{k=-\infty}^{\infty} 2^{k+1} m(E_k) < \infty, \]

and \( f \) is integrable.

(b) Following part a), we need to find the measure of the sets \( E_k \). If \( K \geq 0 \), then

\[ 2^k < |x|^{-a} \leq 2^{k+1}, \]

and

\[ 2^{-k} > |x|^a \geq 2^{-k-1} \]

\[ 2^{-k} > |x| > 2^{-k-1} \]

Then \( m(E_k) = 2 \cdot 2^{-k-1} (2^{\frac{1}{a}} - 1) \). If \( k < 0 \), then \( 2^k < |x|^{-a} \leq 2^{k+1} \) implies \( |x| \geq 1 \), hence \( m(E_k) = 0 \), if \( k < 0 \). Thus,

\[ \sum_{k=-\infty}^{\infty} 2^k m(E_k) = \sum_{k=-\infty}^{\infty} 2^{k+1} \cdot 2^{-k-1} (2^{\frac{1}{a}} - 1) \]

\[ \sum_{k=-\infty}^{\infty} 2^k m(E_k) = (2^{\frac{1}{a}} - 1) \sum_{k=0}^{\infty} 2^{(k+1)(a-1)} \]

\[ \sum_{k=-\infty}^{\infty} 2^k m(E_k) = (2^{\frac{1}{a}} - 1) \sum_{k=0}^{\infty} \left[ 2^{(a-1)(k+1)} \right] \\
\]

Notice that this geometric series converge if and only if \( 2^{(a-1)} < 1 \), and this happens if and only if \( a < 1 \). \( \Box \)