MAT324: Real Analysis – Fall 2014
Assignment 4

Due Thursday, October 6, in class.

Problem 1: Let \( \{f_n\} \) be a sequence of measurable functions defined on \( \mathbb{R} \). Show that the sets

\[
E_1 = \{ x \in \mathbb{R} \mid \lim_{n \to \infty} f_n(x) \text{ exists and is finite} \}
\]

\[
E_2 = \{ x \in \mathbb{R} \mid \lim_{n \to \infty} f_n(x) = \infty \}
\]

\[
E_3 = \{ x \in \mathbb{R} \mid \lim_{n \to \infty} f_n(x) = -\infty \}
\]

are measurable.

Problem 2: Let \( C \subset [0, 1] \) be the Cantor middle-thirds set. Suppose that \( f : [0, 1] \to \mathbb{R} \) is defined by \( f(x) = 0 \) for \( x \in C \) and \( f(x) = k \) for all \( x \) in each interval of length \( 3^{-k} \) which has been removed from \( [0, 1] \) at the \( k^{th} \) step of the construction of the Cantor set. Show that \( f \) is measurable and calculate \( \int_{[0,1]} f \, dm \).

Problem 3: Let \( E \) be a measurable set. For a function \( f : E \to \mathbb{R} \) we define the positive part \( f^+ : E \to \mathbb{R} \), \( f^+(x) = \max(f(x), 0) \), and the negative part \( f^- : E \to \mathbb{R} \), \( f^-(x) = \min(f(x), 0) \). Prove that \( f \) is measurable if and only if both \( f^+ \) and \( f^- \) are measurable.

Problem 4: Prove that if \( f \) is integrable on \( \mathbb{R} \) and \( \int_E f(x) \, dm \geq 0 \) for every measurable set \( E \), then \( f(x) \geq 0 \) a.e. \( x \).

Hint: Show that the set \( F = \{ x \mid f(x) < 0 \} \) is null.

Problem 5: Let \( E \) be a measurable set. Suppose \( f \geq 0 \) and let \( E_k = \{ x \in E \mid 2^k < f(x) \leq 2^{k+1} \} \) for any integer \( k \). If \( f \) is finite almost everywhere, then

\[
\bigcup_{k=-\infty}^{\infty} E_k = \{ x \in E \mid f(x) > 0 \},
\]

and the sets \( E_k \) are disjoint.

(a) Prove that \( f \) is integrable if and only if \( \sum_{k=-\infty}^{\infty} 2^k m(E_k) < \infty \).

(b) Let \( a > 0 \) and consider the function

\[
f(x) = \begin{cases} 
|x|^{-a} & \text{if } |x| \leq 1 \\
0 & \text{otherwise}.
\end{cases}
\]

Use part a) to show that \( f \) is integrable on \( \mathbb{R} \) if and only if \( a < 1 \).