Linear Series on Algebraic Varieties

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In recent years, ideas and methods from the theory of vector bundles have been applied to study some classical sorts of questions concerning linear series on curves, surfaces and other algebraic varieties. Our purpose here is to survey some of the problems to which these and related techniques have been applied.

Many of the results we describe in one way or another involve the equations defining projective varieties. There are several motivations for studying questions along these lines. The first is historical—the equations defining varieties have been of interest to geometers at least since Noether, and it is natural to try to clarify and extend classical results as much as possible. Secondly, there is a "practical" motivation. It has become common for geometers to use computers to analyze explicit varieties, and for the most part the only way to describe a variety to a computer is by giving its equations. Finally, it seems likely that vector bundle methods will play an increasingly important role in the study of linear series, and the classical questions we consider here have proven to be good testing grounds for these techniques.

Limitations of space preclude more than a cursory discussion of methods of proof. We refer to [L3] for an overview of some of these. However we have attempted to survey some of the many interesting open problems in this area. We work throughout over the complex numbers $\mathbb{C}$.

§ 1. Castelnuovo-Mumford Regularity

Let $X \subset \mathbb{P}^r$ be an irreducible variety of dimension n, and denote by $\mathscr{I}_X$ the ideal sheaf of X in $\mathbb{P}^r$. Recall that one says that X is *k-regular* if $H^i(\mathbb{P}^r, \mathscr{I}_X(k-i)) = 0$ for $i \geq 1$ ([M2], [M3]); the *regularity* $\operatorname{reg}(X)$ of $X \subset \mathbb{P}^r$ is the least such k. The interest in this concept stems partly from the fact that $\operatorname{reg}(X)$ governs the complexity of computing the syzygies and other invariants of X ([BS1, BS2]). For example, a theorem of Mumford (cf. [EG]) states that X is k-regular if and only if for every $p \geq 0$, the minimal generators of the kth module of syzygies of the homogeneous

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ideal I_X occur in degrees $\leq k + p$. This has led to a substantial body of work aimed at bounding the regularity of X in terms of its invariants or (more recently) the degrees of its defining equations. In each case there is a fascinating tension between the situation for arbitrary schemes $X \subset \mathbb{P}^r$ — for which results and examples show that the regularity can be very large — and the geometric situation in which X is smooth or at least reduced, where one expects much stronger statements.

We start with the *Castelnuovo problem* of bounding the regularity of X in terms of its invariants. For an arbitrary scheme $X \subset \mathbb{P}^r$, Gotzmann [Gotz, G3] has given a best-possible bound in terms of the Hilbert polynomial of X. For instance, if $X \subset \mathbb{P}^r$ is a curve of degree d and arithmetic genus g, his result states that $\operatorname{reg}(X) \leq \binom{d-1}{2} + d - g$. The question now arises if one can do better assuming X is smooth or reduced.

In practice the main difficulty is to control $H^1(\mathbb{P}^r, \mathscr{I}_X(k))$, which is the classical question of whether hypersurfaces of degree k trace out a complete linear series on X. A theorem of Castelnuovo asserts that when $X \subset \mathbb{P}^r$ ($r \geq 3$) is a smooth curve of degree d, then $H^1(\mathbb{P}^r, \mathscr{I}_X(k)) = 0$ for $k \geq d - 2$, and consequently X is $(d-1)$-regular. This is optimal for curves in $\mathbb{P}^3$, but not in general. In fact, Castelnuovo's result was completed in [GLP], where vector bundle techniques are used to show that if $X \subset \mathbb{P}^r$ is a reduced irreducible non-degenerate curve of degree d, then X is $(d + 2 - r)$-regular. Furthermore the borderline examples are classified: roughly speaking, X fails to be $(d + 1 - r)$-regular if and only if it has a $(d + 1 + r)$-secant line. This possibility of giving geometric explanations for extremal algebraic behavior is one of the interesting themes in this circle of ideas; another example appears in [G3].

These results for curves have led various people to hope that if $X \subset \mathbb{P}^r$ is a smooth non-degenerate complex projective variety of degree d and dimension n, then X is $(d + n + 1 - r)$-regular [EG]. At least when $r \geq 2n + 1$, this would be the best possible linear inequality. It is proved for surfaces in [P] and [L1], and for 3-folds in [R] when $r \geq 9$. In higher dimensions, the situation is much less clear. There is a rather weak bound of Mumford's (cf. [BEL, (2.1)]), but new ideas are apparently needed to treat the problem in general. It would already be very interesting to show that X is d-regular. There is also no clear understanding of what to expect if X is singular (but reduced and irreducible). For an arbitrary scheme $X \subset \mathbb{P}^r$, Ravi [Rav] asks whether one has the inequality $\operatorname{reg}(X_{\mathrm{red}}) \leq \operatorname{reg}(X)$; this may be too much to wish for, but it would be nice to settle the matter.

As we have noted, attempts to bound the regularity of a projective variety $X \subset \mathbb{P}^r$ are motivated in part by the desire to bound the complexity of computing the syzygies and related invariants of X. Bayer has remarked that it is then natural to ask for bounds in terms of the degrees of the defining equations of X – which are usually primary data when one sets out to make an actual computation – rather than the degree of X, which may not be so obvious in a given situation. For an arbitrary scheme, this regularity can again be horrendously large: there are examples due to Mayr-Meyer-Bayer-Stillman [BS1] of schemes $X \subset \mathbb{P}^r$ defined by hypersurfaces of degree d with $\operatorname{reg}(X) \geq (d-2)^{2^{(r/10)}}$. However when X is smooth, the regularity grows much more slowly with d and r. In fact, it comes as a pleasant surprise to note that elementary arguments using the Kodaira vanishing theorem lead to the optimal bound in this case:

Theorem 1.1 [BEL]. *Assume that $X \subset \mathbb{P}^r$ is a smooth variety of dimension n and codimension e defined by hypersurfaces of degrees $d_1 \geq d_2 \geq \cdots \geq d_m$. Then X is $(d_1 + \cdots + d_e - e + 1)$-regular, and X fails to be $(d_1 + \cdots + d_e - e)$-regular if and only if it is the complete intersection of hypersurfaces of degrees $d_1, d_2, \ldots, d_e$.*

Note that only the degrees of the first $e = \operatorname{codim}(X, \mathbb{P}^r)$ defining equations come into play here. It would be very interesting to know to what extent one can allow X to be singular.

§ 2. Syzygies of Algebraic Varieties

It is useful to think of the results described in the previous section as being *extrinsic* in nature, in the sense that they refer to a given projective embedding $X \subset \mathbb{P}^r$, with the main difficulties coming from the fact that X might not be linearly normal. But one can also take a more *intrinsic* point of view, where one deals with the embedding $X \subset \mathbb{P} = \mathbb{P}(H^0(L))$ defined by the complete linear system associated to a very ample line bundle L on X. Here it is usually not so hard to compute $\operatorname{reg}(X)$, so one can ask for more precise information. A very interesting line of inquiry – inaugurated by Green in [G1] – is to study the syzygies of X.

We start with some notation. Let L be a very ample line bundle on a projective variety X, defining an embedding $X \subset \mathbb{P} = \mathbb{P}(H^0(L))$. Denote by $S = \operatorname{Sym}^{\bullet} H^0(L)$ the homogeneous coordinate ring of the projective space $\mathbb{P}$, and consider the graded S-module $R = R(L) = \oplus H^0(X, L^d)$. Let $E_{\bullet} = E_{\bullet}(L)$ be a minimal graded free resolution of R:

$$\begin{array}{ccccccccc} \cdots \longrightarrow & \oplus S(-a_{2,j}) & \longrightarrow & \oplus S(-a_{1,j}) & \longrightarrow & \begin{array}{c} S \\ \oplus \\ \oplus S(-a_{0,j}) \end{array} & \longrightarrow & R & \longrightarrow 0. \\ & \| & & \| & & \| & & & \\ & E_2 & & E_1 & & E_0 & & & \end{array}$$

We have indicated here the fact that R has a canonical generator in degree zero. It is easy to see that all $a_{0,j} \geq 2$ and all $a_{i,j} \geq i + 1$ when $i \geq 1$. To extend classical results on normal generation and presentation we ask when the first few E_i are as simple as possible:

Definition 2.1. The line bundle L saisfies *property* (N_p) if

$$E_0 = S \text{ when } p \geq 0$$

and

$$E_i = \oplus S(-i-1) \text{ [i.e. all } a_{i,j} = i+1] \qquad \text{for} \qquad 1 \leq i \leq p.$$

Note that if $E_0 = S$, then $E.$ determines a resolution of the homogeneous ideal $I = I_{X/\mathbb{P}}$ of X in $\mathbb{P}(H^0(L))$. Thus the definition may be summarized very concretely as follows:

L satisfies (N_0) $\Leftrightarrow$ X embeds in $\mathbb{P}(H^0(L))$ as a projectively normal variety;

L satisfies (N_1) $\Leftrightarrow$ (N_0) holds for L, and the homogeneous ideal I_X of X is generated by quadrics;

L satisfies (N_2) $\Leftrightarrow$ (N_0) and (N_1) hold for L, and the module of syzygies among quadratic generators $Q_i \in I$ is spanned by relations of the form

$$\sum L_i Q_i = 0,$$

where the L_i are *linear* polynomials;

and so on. Properties (N_0) and (N_1) are what Mumford [M3] calls respectively *normal generation* and *normal presentation.*

A classical theorem of Castelnuovo, Mattuck and Mumford states that if X is a smooth curve of genus g, and if $\deg(L) \geq 2g + 1$, then L is normally generated. Fujita and St. Donat proved that if $\deg(L) \geq 2g + 2$, then L satisfies (N_1). These results were extended and clarified by Green [G1], who showed that if $\deg(L) \geq 2g + 1 + p$, then L satisfies (N_p). Green's theorem was recovered as a consequence of an analogous statement of finite sets in [GL3], where it was also proved that if $\deg(L) = 2g + p$, then L fails to satisfy (N_p) if and only if either X is hyperelliptic or Φ_L embeds X with a $(p + 2)$-secant p-plane.

The result just quoted gives a first indication of the fact that (at least conjecturally) the syzygies of a smooth curve X are intimately connected with its geometry. The crucial invariant here is the *Clifford index* Cliff(X) of X. Referring for instance to [GL1] for the precise definition, suffice it to say that Cliff(X) is a non-negative integer which measures from the point of view of special divisors how general X is in moduli. For instance Cliff(X) $= 0$ if and only if X is hyperelliptic, and Cliff(X) $= 1$ if and only if X is trigonal or a smooth plane quintic. One has the inequality $0 \leq \mathrm{Cliff}(X) \leq [(g - 1)/2]$, and $\mathrm{Cliff}(X) = [(g - 1)/2]$ when X is a general curve of genus g.

The hope is that uniform results on the syzygies of a curve X can be strengthened to take into account its Clifford index. The clearest expression of this philosophy which has actually been proved to date is a theorem of [GL1] to the effect that if L is a very ample line bundle on X with $\deg(L) \geq 2g + 1 - 2 \cdot h^1(L) - \mathrm{Cliff}(X)$, then L is normally generated. This includes the result of Castelnuovo et al. mentioned above, and for example taking L to be the canonical bundle Ω it yields Noether's celebrated theorem that Ω is normally generated unless X is hyperelliptic. One conjectures [GL1] that if L is a very ample line bundle with $\deg(L) \geq 2g + 1 + p - 2 \cdot h^1(L) - \mathrm{Cliff}(X)$, then L satisfies (N_p) provided that Φ_L does not embed X with a $(p + 2)$-secant p-plane. This would imply all the known results on syzygies of curves. But its most interesting consequence would be a beautiful conjecture of Green's on the syzygies of canonical curves:

Conjecture 2.2 [G1]. *Cliff(X) is the least integer p for which (N_p) fails for the canonical bundle Ω.*

Green's conjecture has already sparked a considerable amount of work. For instance, drawing on some ideas of Chang and Ran as well as some computer computations by Bayer and Stillman, Ein [E] proves (2.2) on a general curve for $p \leq 3$. The most striking progress to date is due to Voisin [V1] and Schreyer [S], who verify the first non-classical case $p = 2$. Schreyer also gives examples to show that (2.2) fails in characteristic 2. A possible approach to this conjecture is suggested in [PR]. In a somewhat different direction, one has

Conjecture 2.3 ([GL1], Conj. 3.7). *One can read off the "gonality" of a curve X from the grading of the resolution $E_\bullet = E_\bullet(L)$ of any one line bundle of sufficiently large degree.*

In fact, one expects the gonality to be determined by the tail end of $E_\bullet$. One hopes that (2.3) should be easier to establish than (2.2).

Results on normal generation and presentation of curves have traditionally gone hand in hand with analogous results for abelian varieties. Suppose then that X is an abelian variety of dimension n, and let L be an ample line bundle on X. A theorem of Mumford [M1, M3], Koizumi [K] and Sekiguchi [Sek] states that $L^{\otimes k}$ is normally generated provided that $k \geq 3$, and Mumford [M3] proved that X is scheme-theoretically cut out by quadrics under the embedding defined by $L^{\otimes k}$ provided that $k \geq 4$. By analogy with the situation on curves, it is then natural to conjecture [L3] that if $k \geq p + 3$, then $L^{\otimes k}$ satisfies (N_p). Kempf [Kmf] has recently made considerable progress on this conjecture. Specifically, he proves if $p \geq 1$ then (N_p) holds for $L^{\otimes k}$ so long as $k \geq 2p + 2$. It would be wonderful to prove the conjecture in general.

Finally, what can one say in higher dimensions? Green [G2] proved that on an arbitrary smooth variety X of dimension n, any sufficiently positive line bundle L satisfies (N_p). But naturally one would like an explicit result. Mukai observed that the known theorems deal with embeddings defined by bundles of the type $K_X \otimes P$, where P is an explicit multiple of a suitably positive bundle. He suggested that in general one should aim for statements having this shape. In this direction one has

Theorem 2.4 [EL]. *Let A be a very ample line bundle on X, and let B be a numerically effective bundle on X. If $k \geq n + 1 + p$, then $K_X \otimes A^{\otimes k} \otimes B$ satisfies (N_p).*

So for example, $K_X \otimes A^{\otimes n+1}$ is normally generated (which has a quick proof: cf. [BEL], [AS] and [ABS]), and in the embedding defined by $K_X \otimes A^{\otimes n+2}$ the homogeneous ideal I_X is generated by quadrics. It would be very interesting to prove analagous statements assuming only that A is ample. Butler [But] shows that if X is a projective bundle over a curve, and if A is an ample line bundle on X, then $K_X \otimes A^{\otimes 2n+1}$ is normally generated. A rash optimist might be tempted to speculate that $K_X \otimes A^{\otimes n+2+p}$ satisfies (N_p) whenever A is an ample line bundle on an arbitrary smooth variety X. However even when $n = 3$ it is unknown whether $K_X \otimes A^{\otimes n+2}$ is very ample (this is a celebrated conjecture of Fujita), so that the moment it seems premature to hope for statements of this type for syzygies. However as Mukai points out, it might be reasonable to ask for sharp theorems for surfaces. When $X = \mathbb{P}^n$, results on syzygies and related issues amount to statements about multiplicative properties of subspaces of the polynomial ring. Green [G2] has obtained some general theorems along these lines, for which he has given interesting applications to the Hodge theory of hypersurfaces. We refer the reader to [G4] for a survey.

§ 3. Linear Series and Vector Bundles on Surfaces

As we remarked in the Introduction, most of the results described above are proved using vector bundle techniques. For questions involving regularity and syzygies, the arguments are largely cohomological in nature. However on surfaces, bundles have been used in a more geometrical fashion to study linear series. In this final section we survey two such applications. Henceforth X denotes a smooth complex projective surface.

Consider a curve C on X, and a line bundle A on C which is generated by its global sections. Thinking of A as a coherent sheaf on X, there is a natural surjective evaluation map $e_{C,A}: H^0(A) \otimes_{\mathbb{C}} \mathcal{O}_X \to A$. Set $F = F_{C,A} = \ker e_{C,A}$; thus F is a vector bundle of rank $h^0(C, A)$ on X. Philosophically, F encodes information about the pair (C, A) into a geometric object that lives globally on X. When X is a K3 surface, an analysis of these bundles together with a theorem of Mukai's [Mk1, Mk2] on the smoothness of the moduli space of simple bundles on X leads to the following

Theorem 3.1 [L2]. *Let X be a* K3 *surface, and let $C_0 \subset X$ be a smooth curve having the property that every curve in the linear series $|C_0|$ is reduced and irreducible. Then the general member $C \in |C_0|$ behaves generically in the sense of Brill-Noether theory, i.e. the varieties $W_d^r(C)$ of special linear series on C have the postulated dimensions and $W_d^r(C)$ is smooth away from $W_d^{r+1}(C)$.*

We refer for instance to [ACGH] for a fuller discussion of the Brill-Noether-Petri package. The hypothesis is satisfied for instance when $\mathrm{Pic}(X) = \mathbb{Z} \cdot [C_0]$, and the theorem then leads to a very quick proof of an important result of Gieseker's [Gies] concerning the behavior of the varieties $W_d^r(C)$ on a general curve of genus g. The bundles $F_{C,A}$ were studied on an arbitrary surface by Tjurin [T]. They also lead to the proof [GL2] of a conjecture of Harris, Mumford and Green to the effect that all curves in a given linear series on a K3 surface have the same Clifford index. Related results appear in [DM] and [Par]. Reid [Reid] has some intriguing ideas about how bundles of this type might be relevant to the study of canonical surfaces. More recently, Voisin [V2] has used an infinitesimal analogue of the $F_{C,A}$ to prove a very beautiful result about the Wahl map $\gamma_\Omega: \bigwedge^2 H^0(C, \Omega) \to H^0(C, \Omega^3)$ on a smooth curve C of genus g.

Finally, no survey of vector bundles and linear series would be complete without mentioning the very interesting work of Igor Reider [Rdr] on linear series on a smooth projective surface X. Suppose that L is a line bundle on X such that $K_X \otimes L$ fails to be globally generated at a point $x \in X$. Then $H^1(X, \mathscr{I}_X \otimes K_X \otimes L) \neq 0$, and so by Grothendieck duality and a well-known construction of Serre's one can construct a non-split extension of the form $0 \to \mathcal{O}_X \to E \to L \to 0$. Reider's idea — which has its antecedents in a proof by Mumford of the Ramanujam vanishing theorem — is that suitable numerical conditions on L will force E to be Bogomolov-unstable. Analyzing this instability geometrically then leads to the following

Theorem 3.2 [Rdr]. *Assume that L is a nef line bundle on X such that $K_X \otimes L$ is not globally generated. If $c_1(L^2) \geq 5$, then there is an effective divisor $D \subset X$ such that either $c_1(L) \cdot D = 0$ and $D^2 = -1$, or $c_1(L) \cdot D = 1$ and $D^2 = 0$.*

Reider proves an analogous statement for the failure of $K_X \otimes L$ to be very ample. This theorem has a surprising range of applications, among them a quick proof of Bombieri's results on pluricanonical maps of surfaces. Reider's theorem has attracted a lot of attention, and it has been extended in several directions. We refer to [Kot] and the conference proceedings [Som] for a sampling of some of the work in this direction.

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