Math 122 (Fall ’12)
Sample Questions for Midterm 2

1. (20pts) Find the derivatives for the following functions

1. \( x^4 + 5x^3 - 2x^2 + 5 \)
   Solutions: \((x^4 + 5x^3 - 2x^2 + 5)' = 4x^3 + 15x^2 - 4x.\)

2. \( x^{100} + e^{100} \)
   Solutions: \((x^{100} + e^{100})' = (x^{100})' + (e^{100})' = 100x^{99}.\) (Note that \(e^{100}\) is a constant.)

3. \( \sqrt[3]{x} - \frac{1}{\sqrt[3]{x^2}} \)
   Solutions: \((\sqrt[3]{x} - \frac{1}{\sqrt[3]{x^2}})' = (\sqrt[3]{x})' - \left(\frac{1}{\sqrt[3]{x^2}}\right)' = (x^{\frac{1}{3}})' - (x^{-\frac{2}{3}})' = \frac{1}{3}x^{-\frac{2}{3}} + \frac{2}{3x^{\frac{5}{3}}}.\)

4. \( e^t + \ln t \)
   Solutions: \((e^t + \ln t)' = (e^t)' + (\ln t)' = e^t + \frac{1}{t}.\)

5. \( e^t \cdot \ln t \)
   Solutions: By product rule, \((e^t \cdot \ln t)' = (e^t)' \cdot \ln t + e^t \cdot (\ln t)' = e^t \cdot \ln t + e^t \cdot \frac{1}{t}.\)

6. \( e^{u^3+u+2} \)
   Solutions: By chain rule, \((e^{u^3+u+2})' = e^{u^3+u+2} \cdot (u^3 + u + 2)' = e^{u^3+u+2} \cdot (3u^2 + 1).\)
7. $\sqrt{\ln x + 2}$
Solutions: By chain rule, $(\sqrt{\ln x + 2})' = [(\ln x + 2)^{\frac{1}{2}}]' = \frac{1}{2} \cdot (\ln x + 2)^{-\frac{1}{2}} \cdot \frac{1}{x} = \frac{1}{2x} \cdot (\ln x + 2)^{-\frac{1}{2}}.

8. $s^3 \cdot \ln(e^s + e^{-s})$
Solutions: $[s^3 \cdot \ln(e^s + e^{-s})]' = (s^3)' \cdot \ln(e^s + e^{-s}) + s^3 \cdot [\ln(e^s + e^{-s})]' = 3s^2 \cdot \ln(e^s + e^{-s}) + s^3 \cdot \frac{1}{e^s + e^{-s}} \cdot (e^s + e^{-s})' = 3s^2 \cdot \ln(e^s + e^{-s}) + s^3 \cdot \frac{1}{e^s + e^{-s}} \cdot (e^s - e^{-s}).$

9. $\frac{x^2 - 1}{x^2 + 1}$
Solutions: By quotient rule, $(\frac{x^2 - 1}{x^2 + 1})' = \frac{(x^2 - 1)'(x^2 + 1) - (x^2 + 1)'(x^2 - 1)}{(x^2 + 1)^2} = \frac{4x}{(x^2 + 1)^2}.$

10. $x^x$ (Hint: $x = e^{\ln x}$, and thus $x^x = e^{\ln x \cdot x}$)
Solutions: $(x^x)' = (e^{\ln x \cdot x})' = e^{\ln x \cdot x} \cdot (\ln x \cdot x)' = e^{\ln x \cdot x} \cdot [\ln(x)' \cdot x + \ln x \cdot (x)'] = e^{\ln x \cdot x} \cdot (\frac{1}{x} \cdot x + \ln x) = e^{\ln x \cdot x} \cdot (1 + \ln x) = x^x \cdot (1 + \ln x).$
2. (10pts) Find the equation of the tangent line to the graph of \( y = \ln x \) at \( x = e \). Graph the function and the tangent line on the same axes.

Solutions: Firstly, let us recall that the equation of tangent line of \( y = f(x) \) at \( x = a \) is
\[
y = f'(a)(x - a) + f(a).
\]
Now the function is \( y = f(x) = \ln x \) and we try to write down the equation of tangent line at \( x = e \) (i.e. \( a = e \) in the formula). It is easy to see that \( f(e) = \ln(e) = 1 \). To compute \( f'(e) \), we firstly compute the formula of the deriative function: \( y = f'(x) = (\ln x)' = \frac{1}{x} \). Now just plug \( x = e \) into the formula of \( f'(x) \): \( f'(e) = \frac{1}{e} \). So the equation of tangent line of \( y = \ln x \) at \( x = e \) is
\[
y = \frac{1}{e}(x - e) + 1.
\]
It is not difficult to simply it to get
\[
y = e x.
\]
3. (10pts) With length, \( l \), in meters, the period \( T \), in seconds, of a pendulum is given by
\[
T = 2\pi \sqrt{\frac{l}{9.8}}.
\]
a) How fast does the period increase as \( l \) increases? What are units for the rate of change?

b) Does this rate of change increases or decreases as \( l \) increases?

Solutions:

a) The instantaneous rate of change (i.e. how fast) is the derivative \( T'(l) \).
Indeed, \( T'(l) = (2\pi \sqrt{\frac{l}{9.8}})' = (2\pi \cdot \frac{1}{\sqrt{9.8}} \cdot \sqrt{l})' = 2\pi \cdot \frac{1}{\sqrt{9.8}} \cdot (\sqrt{l})' = 2\pi \cdot \frac{1}{\sqrt{9.8}} \cdot (l^{\frac{1}{2}})' = 2\pi \cdot \frac{1}{\sqrt{9.8}} \cdot \frac{1}{2} \cdot l^{-\frac{1}{2}} = \frac{\pi}{\sqrt{9.8}} \cdot l^{-\frac{1}{2}} \text{(second/meter)}.

b) The rate \( T'(l) \) decreases as \( l \) increases. In fact, \( T''(l) = \frac{\pi}{\sqrt{9.8}} \cdot l^{-\frac{3}{2}} = \frac{\pi}{\sqrt{9.8}} \cdot \frac{1}{\sqrt{l}} \). As \( l \) increases, the denominator \( \sqrt{l} \) increases, so the whole fraction \( T''(l) \) decreases.
Or we can compute \( T''(l) \). \( T''(l) = (\frac{\pi}{\sqrt{9.8}} \cdot l^{-\frac{1}{2}})' = -\frac{\pi}{2\sqrt{9.8}} \cdot l^{-\frac{3}{2}} = -\frac{\pi}{2\sqrt{9.8}} \cdot \frac{1}{\sqrt{l}} \). The point is \( T''(l) \) is negative, which implies \( T'(l) \) decreases.
4. (10pts) A yam is put in a hot oven, maintained at a constant temperature $200^\circ C$. At time $t = 30$ minutes, the temperature of the yam is $120^\circ C$ and is increasing at an (instantaneous) rate of $2^\circ C/min$. Newton’s law of cooling implies that the temperature at time $t$ is given by

$$T(t) = 200 - ae^{-bt}. \tag{1}$$

Find $a$ and $b$.

Solutions: "At time $t = 30$ minutes, the temperature of the yam is $120^\circ C"$ means $T(30) = 120$, which is $200 - ae^{-30b} = 120$.

"At time $t = 30$ minutes, the temperature of the yam is increasing at an (instantaneous) rate of $2^\circ C/min"$ means $T'(30) = 2$. Again, to get $T'(30)$, we firstly compute $T'(t)$ and then plug in $t = 30$. Since $T'(t) = abe^{-bt}$ (here we view $t$ as a variable and $a, b$ as constant numbers), $T'(30) = abe^{-30b}$. So $T'(30) = 2$ means $abe^{-30b} = 2$.

To sum up, we get two equations about $a$ and $b$ from the problem.

$$200 - ae^{-30b} = 120 \tag{2}$$

and

$$abe^{-30b} = 2. \tag{3}$$

It is not difficult to see that $ae^{-30b} = 80$ from the first equation. So $e^{-30b} = \frac{80}{a}$. Now we plug this into the second equation: $abe^{-30b} = a \cdot b \cdot \frac{80}{a} = 80b = 2$. So $b = \frac{1}{40}$. Plugging this back to either the first or the second equation, we get $a = 80e^{\frac{3}{4}}$.\vspace{10pt}
5. (20pts) Graph the function

\[ f(x) = x^3 - 3x^2 + 2 \]

Your answer should include:

a) Local maxima/minima,

b) Inflection points.
6. (20pts) The derivative of \( f(t) \) is given by \( f'(t) = t^3 - 6t^2 + 8t \) for \( 0 \leq t \leq 5 \).

i) Graph \( f'(t) \), and describe how the function \( f(t) \) changes over the interval \( t \in [0, 5] \).

ii) When is \( f(t) \) increasing and when is it decreasing?

iii) Where does \( f(t) \) have a local maximum and where does it have a local minimum?

iv) What are the inflection points of \( f \)?

Solutions:

i) & ii) The idea is if \( f'(t) > 0 \) (resp. \( f'(t) < 0 \)) on some interval, then \( f(t) \) is increasing (resp. decreasing) on the same interval. Now we determine when will \( f(t) = t^3 - 6t^2 + 8t \) be positive or negative as follows. To start with, let us find out all critical points of \( y = f(t) \) by solving the equation \( f'(t) = 0 \), which is \( t^3 - 6t^2 + 8t = t(t^2 - 6t + 8) = t(t - 2)(t - 4) = 0 \). So we easily get three critical points \( x_1 = 0 \), \( x_2 = 2 \), and \( x_3 = 4 \). These points divide the whole interval \([0, 5]\) into several smaller pieces, and on each piece \( f'(t) \) will have the same sign. Clearly,

- When \( 0 < x < 2 \), \( f'(t) > 0 \), so \( f(t) \) is increasing.
- When \( 2 < x < 4 \), \( f'(t) < 0 \), so \( f(t) \) is decreasing.
- When \( 4 < x < 5 \), \( f'(t) > 0 \), so \( f(t) \) is increasing.

iii) Since our function is defined over a closed subinterval \([0, 5]\), we have to take both critical points \( x = 0, 2, 4 \) and boundary points \( x = 0, 5 \) into consideration. From i) & ii), it is not difficult to see that local maximal points are \( x = 2 \) and \( x = 5 \), and local minimal points are \( x = 0 \) and \( x = 4 \).

iv) To find the inflection points is the same as to solve the equation \( f''(t) = 0 \). Now that \( f'(t) = t^3 - 6t^2 + 8t \), \( f''(t) = 3t^2 - 12t + 8 \). Now let us solve the equation \( 3t^2 - 12t + 8 = 0 \). In general, the solutions of \( ax^2 + bx + c = 0 \) are \( x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \) and \( x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \) (But you’d better try to factor it into two linear terms firstly). Here \( a = 3, b = -12, c = 8 \), so the two solutions or inflection points are \( t_1 = \frac{12 + \sqrt{48}}{6} = \frac{6 + 2\sqrt{3}}{3} \) and \( t_2 = \frac{12 - \sqrt{48}}{6} = \frac{6 - 2\sqrt{3}}{3} \).
7. **(10pts)** When I got up in the morning I put on only a light jacket because, although the temperature was dropping, it seemed that the temperature would not go much lower. But I was wrong. Around noon a northerly wind blew up and the temperature began to drop faster and faster. The worst was around 6pm when, fortunately, the temperature started going back up.

   a) When was there a critical point in the graph of temperature as a function of time?

   b) When was there an inflection point in the graph of temperature as a function of time.

Solutions:

   a) 6pm.

   b) Noon.