Math 122 (Fall '12)

Sample Questions for Midterm 2

1. (20pts) Find the derivatives for the following functions

1. $x^4 + 5x^3 - 2x^2 + 5$
   Solutions: $(x^4 + 5x^3 - 2x^2 + 5)' = 4x^3 + 15x^2 - 4x$.

2. $x^{100} + e^{100}$
   Solutions: $(x^{100} + e^{100})' = (x^{100})' + (e^{100})' = 100x^{99}$. (Note that e^{100} is a constant.)

3. $\sqrt[3]{x} - \frac{1}{\sqrt[3]{x^2}}$
   Solutions: $(\sqrt[3]{x} - \frac{1}{\sqrt[3]{x^2}})' = (\sqrt[3]{x})' - (\frac{1}{\sqrt[3]{x^2}})' = (x^{\frac{1}{3}})' - (x^{-\frac{2}{3}})' = \frac{1}{3}x^{-\frac{2}{3}} + \frac{2}{3}x^{-\frac{5}{3}}$.

4. $e^t + \ln t$
   Solutions: $(e^t + \ln t)' = (e^t)' + (\ln t)' = e^t + \frac{1}{t}$.

5. $e^t \cdot \ln t$
   Solutions: By product rule, $(e^t \cdot \ln t)' = (e^t)' \cdot \ln t + e^t \cdot (\ln t)' = e^t \cdot \ln t + e^t \cdot \frac{1}{t}$.

6. e^{u^3+u+2}
   Solutions: By chain rule, $(e^{u^3+u+2})' = e^{u^3+u+2} \cdot (u^3 + u + 2)' = e^{u^3+u+2} \cdot (3u^2 + 1)$.

7. $\sqrt{\ln x + 2}$
Solutions: By chain rule, $(\sqrt{\ln x + 2})' = [(\ln x + 2)^{\frac{1}{2}}]' = \frac{1}{2} \cdot (\ln x + 2)^{-\frac{1}{2}} \cdot (\ln x + 2)' = \frac{1}{2} \cdot (\ln x + 2)^{-\frac{1}{2}} \cdot \frac{1}{x} = \frac{1}{2x} \cdot (\ln x + 2)^{-\frac{1}{2}}$.

8. $s^3 \cdot \ln(e^s + e^{-s})$
Solutions: $[s^3 \cdot \ln(e^s + e^{-s})]' = (s^3)' \cdot \ln(e^s + e^{-s}) + s^3 \cdot [\ln(e^s + e^{-s})]' = 3s^2 \cdot \ln(e^s + e^{-s}) + s^3 \cdot \frac{1}{e^s + e^{-s}} \cdot (e^s + e^{-s})' = 3s^2 \cdot \ln(e^s + e^{-s}) + s^3 \cdot \frac{1}{e^s + e^{-s}} \cdot (e^s - e^{-s})$.

9. $\frac{x^2-1}{x^2+1}$
Solutions: By quotient rule, $(\frac{x^2-1}{x^2+1})' = \frac{(x^2-1)' \cdot (x^2+1) - (x^2-1) \cdot (x^2+1)'}{(x^2+1)^2} = \frac{(2x) \cdot (x^2+1) - (x^2-1) \cdot (2x)}{(x^2+1)^2} = \frac{4x}{(x^2+1)^2}$.

10. x^x (Hint: $x = e^{\ln x}$, and thus $x^x = e^{\ln x \cdot x}$)
Solutions: $(x^x)' = (e^{\ln x \cdot x})' = e^{\ln x \cdot x} \cdot (\ln x \cdot x)' = e^{\ln x \cdot x} \cdot [(\ln x)' \cdot x + \ln x \cdot (x)'] = e^{\ln x \cdot x} \cdot (\frac{1}{x} \cdot x + \ln x) = e^{\ln x \cdot x} \cdot (1 + \ln x) = x^x \cdot (1 + \ln x)$.

2. (10pts) Find the equation of the tangent line to the graph of $y = \ln x$ at $x = e$. Graph the function and the tangent line on the same axes.

Solutions: Firstly, let us recall that the equation of tangent line of $y = f(x)$ at $x = a$ is

$$y = f'(a)(x - a) + f(a).$$

Now the function is $y = f(x) = \ln x$ and we try to write down the equation of tangent line at $x = e$ (i.e. $a = e$ in the formula). It is easy to see that $f(e) = \ln(e) = 1$. To compute $f'(e)$, we firstly compute the formula of the deriative function: $y = f'(x) = (\ln x)' = \frac{1}{x}$. Now just plug $x = e$ into the formula of $f'(x)$: $f'(e) = \frac{1}{e}$. So the equation of tangent line of $y = \ln x$ at $x = e$ is

$$y = \frac{1}{e}(x - e) + 1.$$

It is not difficult to simply it to get

$$y = \frac{1}{e}x.$$

3. (10pts) With length, l, in meters, the period T, in seconds, of a pendulum is given by

$$T = 2\pi\sqrt{\frac{l}{9.8}}.$$

a) How fast does the period increase as l increases? What are units for the rate of change?

b) Does this rate of change increases or decreases as l increases?

Solutions:

a) The instantaneous rate of change (i.e. how fast) is the deriative $T'(l)$. Indeed, $T'(l) = (2\pi\sqrt{\frac{l}{9.8}})' = (2\pi \cdot \frac{1}{\sqrt{9.8}} \cdot \sqrt{l})' = 2\pi \cdot \frac{1}{\sqrt{9.8}} \cdot (\sqrt{l})' = 2\pi \cdot \frac{1}{\sqrt{9.8}} \cdot (l^{\frac{1}{2}})' = 2\pi \cdot \frac{1}{\sqrt{9.8}} \cdot \frac{1}{2} \cdot l^{-\frac{1}{2}} = \frac{\pi}{\sqrt{9.8}} \cdot l^{-\frac{1}{2}} (second/meter)$.

b) The rate $T'(l)$ decreases as l increases. In fact, $T'(l) = \frac{\pi}{\sqrt{9.8}} \cdot l^{-\frac{1}{2}} = \frac{\pi}{\sqrt{9.8}} \cdot \frac{1}{\sqrt{l}}$. As l increases, the denominator $\sqrt{l}$ increases, so the whole fraction $T'(l)$ decreases.
Or we can compute $T''(l)$. $T''(l) = (\frac{\pi}{\sqrt{9.8}} \cdot l^{-\frac{1}{2}})' = -\frac{\pi}{2\sqrt{9.8}} \cdot l^{-\frac{3}{2}} = -\frac{\pi}{2\sqrt{9.8}} \cdot \frac{1}{\sqrt{l^3}}$. The point is $T''(l)$ is negetive, which implies $T'(l)$ decreases.

4. (10pts) A yam is put in a hot oven, maintained at a constant temperature 200° C. At time $t = 30$ minutes, the temperature of the yam is 120° C and is increasing at an (instantaneous) rate of 2° C/min. Newton's law of cooling implies that the temperature at time t is given by

$$T(t) = 200 - ae^{-bt}.$$

Find a and b.

Solutions: "At time $t = 30$ minutes, the temperature of the yam is 120° C" means $T(30) = 120$, which is $200 - ae^{-30b} = 120$.
"At time $t = 30$ minutes, the temperature of the yam is increasing at an (instantaneous) rate of 2° C/min" means $T'(30) = 2$. Again, to get $T'(30)$, we firstly compute $T'(t)$ and then plug in $t = 30$. Since $T'(t) = abe^{-bt}$ (here we view t as a variable and a, b as constant numbers), $T'(30) = abe^{-30b}$. So $T'(30) = 2$ means $abe^{-30b} = 2$.
To sum up, we get two equations about a and b from the problem.

$$200 - ae^{-30b} = 120$$

and

$$abe^{-30b} = 2.$$

It is not difficult to see that $ae^{-30b} = 80$ from the first equation. So $e^{-30b} = \frac{80}{a}$. Now we plug this into the second equation: $abe^{-30b} = a \cdot b \cdot \frac{80}{a} = 80b = 2$. So $b = \frac{1}{40}$. Pluging this back to either the first or the second equation, we get $a = 80e^{\frac{3}{4}}$.

5. (20pts) Graph the function

$$f(x) = x^3 - 3x^2 + 2$$

Your answer should include:

a) Local maxima/minima,

b) Inflection points.

6. (20pts) The derivative of $f(t)$ is given by $f'(t) = t^3 - 6t^2 + 8t$ for $0 \le t \le 5$.

i) Graph $f'(t)$, and describe how the function $f(t)$ changes over the interval $t \in [0, 5]$.

ii) When is $f(t)$ increasing and when is it decreasing?

iii) Where does $f(t)$ have a local maximum and where does it have a local minimum?

iv) What are the inflection points of f?

Solutions:

i) & ii) The idea is if $f'(t) > 0$ (resp. $f'(t) < 0$) on some interval, then $f(t)$ is increasing (resp. decreasing) on the same interval. Now we determine when will $f(t) = t^3 - 6t^2 + 8t$ be positive or negative as follows. To start with, let us find out all critical points of $y = f(t)$ by solving the equation $f'(t) = 0$, which is $t^3 - 6t^2 + 8t = t(t^2 - 6t + 8) = t(t-2)(t-4) = 0$. So we easily get three critical point $x_1 = 0$, $x_2 = 2$ and $x_3 = 4$. These points divide the whole interval $[0, 5]$ into several smaller pieces, and on each piece $f'(t)$ will have the same sign. Clearly,

- When $0 < x < 2$, $f'(t) > 0$, so $f(t)$ is increasing.
- When $2 < x < 4$, $f'(t) < 0$, so $f(t)$ is decresing.
- When $4 < x < 5$, $f'(t) > 0$, so $f(t)$ is increasing.

iii) Since our function is defined over a closed subinterval $[0, 5]$, we have to take both critical points ($x = 0, 2, 4$) and boundary points ($x = 0, 5$) into consideration. From i) & ii), it is not difficult to see that local maximal points are $x = 2$ and $x = 5$, and local minimal points are $x = 0$ and $x = 4$.

iv) To find the inflection points is the same as to solve the equation $f''(t) = 0$. Now that $f'(t) = t^3 - 6t^2 + 8t$, $f''(t) = 3t^2 - 12t + 8$. Now let us solve the equation $3t^2 - 12t + 8 = 0$. In general, the solutions of $ax^2 + bx + c = 0$ are $x_1 = \frac{-b+\sqrt{b^2-4ac}}{2a}$ and $x_2 = \frac{-b-\sqrt{b^2-4ac}}{2a}$ (But you'd better try to factor it into two linear terms firstly). Here $a = 3, b = -12, c = 8$, so the two solutions or inflection points are $t_1 = \frac{12+\sqrt{48}}{6} = \frac{6+2\sqrt{3}}{3}$ and $t_2 = \frac{12-\sqrt{48}}{6} = \frac{6-2\sqrt{3}}{3}$.

7. (10pts) When I got up in the morning I put on only a light jacket because, although the temperature was dropping, it seemed that the temperature would not go much lower. But I was wrong. Around noon a northerly wind blew up and the temperature began to drop faster and faster. The worst was around 6pm when, fortunately, the temperature started going back up.

a) When was there a critical point in the graph of temperature as a function of time?

b) When was there an inflection point in the graph of temperature as a function of time.

Solutions:

a) 6pm.

b) Noon.