Math 534 (Fall '17)

Review 2

(1) a) Prove that $\langle x^2+1, 3\rangle$ is a prime ideal in $\mathbb{Z}[x]$, while $\langle x^2+1, 2\rangle$ is not.
b) Classify all the prime and maximal ideals in $\mathbb{Z}[x]$.

(2) a) Prove that $x^3+12x^2+20x+6$ is irreducible over $\mathbb{Z}[i]$.
b) Prove that $p(x) = x^4-4x^2+8x+2$ is irreducible over $\mathbb{Q}(\sqrt{-2})$. First, use that $\mathbb{Z}(\sqrt{-2})$ is UFD, to show that $p(x)$ has no root in $\mathbb{Q}(\sqrt{-2})$. Then check that there is no quadratic decomposition.

(3) a) List all ideals in $F[x]/\langle f(x)\rangle$ (where F is a field, f any polynomial).
b) Do the same for the localizations $F[x]_f$ (i.e. all powers of f become invertible) and for f irreducible $F[X]_{\langle f\rangle}$ (i.e. the localization to the prime ideal $\langle f\rangle$).

(4) The aim of this exercise is to prove that $\mathbb{Z}[i]/\langle\alpha\rangle$ is a ring with $N(\alpha)$ elements.
i) First prove this for irreducible (or prime) π. Distinguish two cases $\pi \in \mathbb{Z}$ or not (what are those two case?). The first case (the integral case) is easy. For the second case consider $p = \pi \cdot \bar{\pi}$. By applying the chinese remainder theorem (state it), conclude that

$$\mathbb{Z}[i]/\langle p\rangle \cong \mathbb{Z}[i]/\langle\pi\rangle \times \mathbb{Z}[i]/\langle\bar{\pi}\rangle$$

as rings. It easy to compute the size of $\mathbb{Z}[i]/\langle p\rangle$. Conclude then that $Z[i]/\langle\pi\rangle$ has the right size.
ii) Now prove the statement about powers of π (irreducible). The idea is to consider the sequence of ideals

$$\langle\pi^n\rangle \subset \langle\pi^{n-1}\rangle \subset \cdots \subset \langle\pi\rangle \subset \mathbb{Z}[i]$$

and the successive quotients $\langle \pi^k \rangle / \langle \pi^{k+1} \rangle$. Prove that $\langle \pi^k \rangle / \langle \pi^{k+1} \rangle \cong \mathbb{Z}[i]/\langle \pi \rangle$ as additive groups and then conclude by induction that $\mathbb{Z}[i]/\langle \pi^n \rangle$ has the right size.

iii) Now prove the general case by using the fact that $\mathbb{Z}[i]$ is UFD and the chinese remainder theorem.

(5) i) The first item is intended to explain why we prefer to work with algebraically closed fields in algebraic geometry ($\mathbb{C}$ is better than $\mathbb{R}$, which is better than $\mathbb{Q}$). List all the prime and maximal ideals in $\mathbb{R}[x]$ and $\mathbb{C}[x]$ in a very explicit way. Say something about prime/max. ideals in $\mathbb{Q}[x]$.

ii) Let's consider the ideals of $\mathbb{C}[x, y]$ (think of it as the ring of regular functions on $\mathbb{C}^2$). By algebraic geometry, we know that the maximal ideals correspond to (closed) points $(a, b) \in \mathbb{C}^2$. For each point explicitly identify the corresponding maximal ideal $m_{(a,b)}$ (i.e. the regular functions vanishing at that point). Argue that $m_{(a,b)}$ is indeed maximal.

iii) By algebraic geometry, we know that the prime ideals in $\mathbb{C}[x, y]$ correspond to irreducible subvarieties of $\mathbb{C}^2$. Based on the dimension of the subvariety, we identify 3 cases. List them!

iv) Prove that all the maximal/prime ideals in $\mathbb{C}[x, y]$ are as given in ii and iii. (i.e. in (ii) and (iii) you produced some examples of maximal and prime ideals, now you try to prove that those examples are exhaustive).

(6) Prove that the ring $\mathbb{Z}[\omega]$, where $\omega = \frac{-1+i\sqrt{3}}{2}$, is a PID. Give an explicit classification of $\mathbb{Z}[\omega]$-modules that can be generated by at most 2 generators.

(7) Consider the cusp $C : (y^2 = x^3) \subset \mathbb{A}^2_{\mathbb{C}}$. The ring of regular functions of C is $R(C) = \mathbb{C}[x, y]/\langle y^2 - x^3 \rangle$.

i) Describe the points and the closed points of C (i.e. the prime and maximal ideals in $R(C)$).

ii) Consider now the localizations $R(C)_p$, where p is a prime ideal and the localization is as usual with respect to $S = R(C) \setminus p$. Can you identify these rings (with more concrete isomorphic rings)?

iii) Discuss the PID property of $R(C)$ and $R(C)_p$.

(8) Let R be a commutative ring and M an R-module. Let S be a multiplicative set. Define $S^{-1}M$ in a manner similar to $S^{-1}R$.
   i) Prove that $S^{-1}M$ is an $S^{-1}R$-module.
   ii) If $0 \to M' \to M \to M'' \to 0$ is an exact sequence of R-modules. Prove that $0 \to S^{-1}M' \to S^{-1}M \to S^{-1}M'' \to 0$ is an exact sequence of $S^{-1}R$-module.
   iii) Is it $S^{-1}M$ isomorphic to $M \otimes_R S^{-1}R$? As R-modules? As $S^{-1}R$-modules?
   iv) What do you conclude about $S^{-1}R$ as an R-module?

(9) Let ALG be the category of commutative rings with identity (i.e. there exists 1; morphisms map 1 to 1) [think $\mathbb{Z}$-algebras]. Consider the covariant functor $F : \mathrm{ALG} \to \mathrm{Sets}$, $F(A) = A^*$ that associates to A its units. Give an explicit ring R such that the functor F is isomorphic to the functor $A \to \mathrm{Hom}_{\mathrm{ALG}}(R, A)$ [i.e. R represents F]. Discuss the exactness of the functor F in the category ALG.

(10) Let V be a k-vector space. Assume given B a non-degenerate symmetric bilinear form. (What does it mean?) Interpret B as an isomorphism between V and its dual V^*. Via this isomorphism, for any basis $\{v_1, \dots, v_n\}$ of V there is a well defined dual basis $\{v_1^*, \dots, v_n^*\}$ of V. (What is an alternative definition of this dual basis?)
   i) Let $\Omega_B = \sum v_i \otimes v_i^* \in V \otimes V$.
   ii) Similarly, $Q_B = \sum v_i \cdot v_i^* \in \mathrm{Sym}(V)$.

   Prove that Ω_B and Q_B are independent of the choice of the basis. Can you interpret this in terms of naturally defined maps induced by the bilinear form B.

(11) Let R be a PID. Classify all finitely generated R-modules that are: projective, injective, or flat respectively.

(12) Let

$$A = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}, \quad \text{and} \quad B = \begin{pmatrix} 5 & 2 & -8 & -8 \\ -6 & -3 & 8 & 8 \\ -3 & -1 & 3 & 4 \\ 3 & 1 & -4 & -5 \end{pmatrix}.$$

Since they are matrices defined over the integers, we can interpret them as matrices over any field F. Find their (Jordan) canonical forms and decide if their are similar. Discuss the dependence on the field F.