Math 534 (Fall '17)

Review 1

Class Equation

(1) Prove that any p-group is solvable. (Starting point: the center of a p-group is non-trivial)

(2) Prove the existence of a p-Sylow subgroup.

(3) Prove that any group of order p^2 is abelian.

Group Actions, Automorphism

(4) Let G be a group such that $\mathrm{Aut}(G)$ is cyclic. Prove that G is abelian.

(5) We know that any inner automorphism of G is realized by acting with G on G by conjugation. Prove that we can embed G as a normal subgroup in some group G' such that the morphism $G' \to \mathrm{Aut}(G)$ given by conjugation in G' is surjective.

(6) i) Let H be a non-normal subgroup of index n in G. Show that if $p||H|$ for some prime p with $p \geq n$. Then H cannot be a simple group[1].

ii) Show there is no simple group of order 504.

Sylow's Theorem, Classification

(7) Classify all groups of order 50 and 51 (the easier case) respectively.

(8) Prove that if G is a group of order 231 the $Z(G)$ contains a Sylow 11-subgroup of G and a Sylow 7-subgroup is normal in G.

Tensor products

(9) Let M and N be two flat R modules. Then $M \otimes_R N$ is a flat R-module.

(10) Prove that $\mathbb{Q}$ is a flat $\mathbb{Z}$ module. Show that $\mathbb{Q}/\mathbb{Z}$ is not flat.

(11) Compute:

- $\mathbb{Z}/m \otimes_{\mathbb{Z}} \mathbb{Z}/n$

[1] No typo here, indeed we care about H and not G.

- Assume $p|n$ and $p|m$. Compute $\mathbb{Z}/n \otimes_{\mathbb{Z}/p} \mathbb{Z}/m$
- $\mathbb{Q}/\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n$
- $k[x,y] \otimes_{k[x]} k[x,z]$
- $k[x,y] \otimes_{k} k[x,z]$

(12) Find a necessary and sufficient condition that $\mathbb{Q}[\alpha] \otimes_{\mathbb{Q}} \mathbb{Q}[\alpha]$ is an integral domain.

(13) Prove that $\mathbb{Z}[i] \otimes \mathbb{R} \cong \mathbb{C}$ as rings. (First prove this as $\mathbb{R}$ modules: related question, let α be a an algebraic integer, what is $\mathbb{Z}[\alpha] \otimes \mathbb{R}$ as $\mathbb{R}$ module?

Canonical Forms

(14) Show that if $A^2 = A$ then A is similar to a diagonal matrix which has only 0's and 1's along the diagonal.

(15) Prove that there are no 3×3 matrices A over $\mathbb{Q}$ with $A^8 = I$, but $A^4 \neq I$.

(16) Determine the Jordan canonical form for the $n \times n$ matrix over $\mathbb{F}_p$ whose entries are all equal to 1 (answer depends on $p|n$ or not).

(17) Determine all possible Jordan canonical forms for a linear transformation with characteristic polynomial $(x-2)^3(x-3)^2$.

(18) Prove that if N is a $n \times n$ nilpotent matrix then in fact $N^n = 0$.

(19) Determine necessary and sufficient conditions for a matrix A (over $\mathbb{C}$) to have a square root (i.e. there exists B with $B^2 = A$).