Math 313 (Fall '09)

Midterm 2

November 5

1. (20 pts)
   a) Give the definition of a ring.

   b) Give an example of
      * an integral domain which is not a field
      * a ring which is not an integral domain
      * a ring without unit
      * a non-commutive ring
      * a field which is not an integral domain

2. (20 pts)
   i) Define what is a morphism and give an example.
   ii) Find all the morphisms $\phi : \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$.

3. (15 pts) Let $R = \mathbb{Z}[X]$ and I be the set of all polynomials of with even coefficients.
   i) Prove that I is an ideal.
   ii) Describe the ring R/I. Is I a prime or maximal ideal.
   iii) Is I generated by a single element? Is R/I a polynomial ring (which one)?

4. (15 pts) The goal of this exercise is to get a good understanding of the ring $R = \mathbb{Z}[i]/\langle 3+i \rangle$
   a) How many elements are in R? (Hint: $\hat{i} = ...$ in R) List all the elements of R!

   b) Compute
   $$\widehat{(2+i)} \cdot \widehat{(3-5i)}$$

   c) Is R commutative? Is there a unit in R?

   d) Is R a field, an integral domain? Give an argument!

5. (15 pts) Let $\mathbb{Z}_3[i] = \{a + bi \mid a, b \in \mathbb{Z}_3\}$.
   i) State the first (or fundamental) theorem of isomorphism.

   ii) Use the fundamental theorem of isomorphism to conclude that
   $$\mathbb{Z}_3[i] \cong \mathbb{Z}_3[x]/\langle x^2 + 1\rangle$$
   To prove this follow the following steps by filling in the dots:
   - define a morphism ϕ :.....
     by .....
   - compute Imϕ
   - to compute Kerϕ, we need to do some work. First, we recall that the kernel of a morphism is .....
   - any ideal in $\mathbb{Z}_3[x]$ is principal, i.e. it is generated by .....
   - a generator of an ideal $I \subset \mathbb{Z}_3[x]$ is characterized by the fact it has ..... degree
   - $x^2 + 1$ has the property that .....
   - at the same time no polynomial of degree .....
   - Conclude!

6. (15 pts) Construct a field with 125 elements. State the various results that you are using for this construction. What is the characteristic of this field?