Math 313 (Fall '09)

Midterm 1

October 8

Note: You have 80 minutes, or about 10 minutes per exercise. The first 4 exercises are worth 15 points, the other 4 are 10 points each.

1. (15 pts)

   a) Give the definition of a group. Give example of a group and another example that fails to be a group (Justify!).

   b) Give examples of 3 non-isomorphic groups of order 18. Explain why your examples are non-isomorphic.

2. (15 pts) Let

$$G = \left\{ X = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \mid \text{ with } a, b \in \mathbb{R} \text{ and not simultaneously } 0 \right\}$$

i) Check that G with matrix multiplication is a group.
ii) Check that G is isomorphic to $\mathbb{C}^*$ (recall $\mathbb{C}^*$ are the non-zero complex numbers). What is the operation on $\mathbb{C}^*$ that makes the isomorphism work?

3. (15 pts) Count the number of elements of order 6 in

   i) $G = S_6$;

   ii) $G = Z_{12}$;

   iii) $G = Z_2 \oplus Z_6$.

   Do the same for the number of subgroups of order 6!

   Hint: there is an easy solution for the second part, once you did the first part.

4. (15 pts)

a) Give an example of group G and 2 subgroups H_1 and H_2 such that H_1 is normal and H_2 is not normal. Justify!

b) Prove that any subgroup of an abelian group is normal

Extra points (5pts):

c) Prove that a subgroup H of index 2 in any group G is normal.

5. (10 pts) The set $G = \{1, 7, 17, 23, 49, 55, 65, 71\}$ is a group under multiplication modulo 96. To which well-known group is G isomorphic to?

6. (10 pts) Find all the subgroups of order 4 of $\mathbb{Z}_8 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_2$.

7. (10 pts) Prove that a group with 11 elements is cyclic. Generalize!
   Note: You are allowed to use L... Theorem, but nothing else.

8. (10 pts) In the list: $\mathbb{Z}_{12}$, $\mathbb{Z}_6 \oplus \mathbb{Z}_2$, $S_3 \oplus \mathbb{Z}_2$, A_4, D_6 there is a repetition. Decide which two groups are isomorphic. Explain!