Math 313 (Fall ’09)
Midterm 1
October 8

Note: You have 80 minutes, or about 10 minutes per exercise. The first 4 exercises are worth 15 points, the other 4 are 10 points each.

1. (15 pts)
   a) Give the definition of a group. Give example of a group and another example that fails to be a group (Justify!).

   b) Give examples of 3 non-isomorphic groups of order 18. Explain why your examples are non-isomorphic.
2. (15 pts) Let

\[ G = \left\{ X = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \mid \text{with } a, b \in \mathbb{R} \text{ and not simultaneously } 0 \right\} \]

i) Check that \( G \) with matrix multiplication is a group.

ii) Check that \( G \) is isomorphic to \( \mathbb{C}^* \) (recall \( \mathbb{C}^* \) are the non-zero complex numbers). What is the operation on \( \mathbb{C}^* \) that makes the isomorphism work?
3. (15 pts) Count the number of elements of order 6 in
   i) $G = S_6$;
   ii) $G = Z_{12}$;
   iii) $G = Z_2 \oplus Z_6$.
Do the same for the number of subgroups of order 6!

**Hint:** there is an easy solution for the second part, once you
did the first part.
4. (15 pts)
   a) Give an example of group $G$ and 2 subgroups $H_1$ and $H_2$ such that $H_1$ is normal and $H_2$ is not normal. Justify!

   b) Prove that any subgroup of an abelian group is normal

Extra points (5pts):
   c) Prove that a subgroup $H$ of index 2 in any group $G$ is normal.
5. (10 pts) The set $G = \{1, 7, 17, 23, 49, 55, 65, 71\}$ is a group under multiplication modulo 96. To which well-known group is $G$ isomorphic to?
6. (10 pts) Find all the subgroups of order 4 of $\mathbb{Z}_8 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_2$. 
7. (10 pts) Prove that a group with 11 elements is cyclic. Generalize!

Note: You are allowed to use L... Theorem, but nothing else.
8. (10 pts) In the list: $\mathbb{Z}_{12}, \mathbb{Z}_6 \oplus \mathbb{Z}_2, S_3 \oplus \mathbb{Z}_2, A_4, D_6$ there is a repetition. Decide which two groups are isomorphic. Explain!