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Maximal Cohen–Macaulay modules over the cone of an elliptic curve

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Abstract

Let $R = k[Y_1, Y_2, Y_3]/(f)$, $f = Y_1^3 + Y_2^3 + Y_3^3$, where k is an algebraically closed field with $\operatorname{char} k \neq 3$. Using Atiyah bundle classification over elliptic curves we describe the matrix factorizations of the graded, indecomposable reflexive R-modules, equivalently we describe explicitly the indecomposable bundles over the projective curve $V(f) \subset \mathbb{P}_k^2$. Using the fact that over the completion $\widehat{R}$ of R every reflexive module is gradable, we obtain a description of the maximal Cohen–Macaulay modules over $\widehat{R} = k[\![Y_1, Y_2, Y_3]\!]/(f)$.

Introduction

The nice classification of vector bundles over elliptic curves obtained by Atiyah [1] enabled C. Kahn to give a description (see [2]) of graded reflexive modules over minimally elliptic singularities in characteristic 0. Kahn was able to describe the Auslander–Reiten quivers of graded reflexive R-modules. But how explicit is Kahn's description in the hypersurface case? According to [3]

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each graded reflexive R-module is given by a matrix factorization. The matrix factorizations are not given in Kahn's paper.

The purpose of our paper is the classification of maximal Cohen–Macaulay modules over the local ring $\widehat{R} = k[\![Y_1, Y_2, Y_3]\!]/(f)$, $f = Y_1^3 + Y_2^3 + Y_3^3$, k an algebraically closed field of characteristic $\neq 3$. According to a result of Kahn (cf. [4, Proposition 5.23]) the k^*-action on $\widehat{R}$ induces an action on the reflexive modules over $\widehat{R}$ and, therefore, a grading on such a module compatible to the k^*-action on $\widehat{R}$. This implies that every reflexive module on $\widehat{R}$ is gradable (in the sense of Yoshino (cf. [5]), that is, for every maximal Cohen–Macaulay $\widehat{R}$-module $\overline{M}$ there exists a graded R-module M such that $\widehat{M} \cong \overline{M}$.

Let M and N be graded indecomposable maximal Cohen–Macaulay R-modules, then $\widehat{M} \cong \widehat{N}$ if and only if $M \cong N(r)$ for some r. We are, therefore, interested in classifying the equivalence classes of graded reflexive modules with two R-modules M and N being equivalent if $M \cong N(r)$ for some s. Therefore, we shall show that Atiyah's classification can be explicitly done for the case of the projective curve $f = Y_1^3 + Y_2^3 + Y_3^3$. We were able to write canonical normal forms for the matrix factorizations of all graded reflexive R-modules of rank one (see Section 3) and to show effectively how we can produce the indecomposable graded reflexive R-modules of ranks $\geqslant 2$ using SINGULAR with help of a computer (see Section 5).

One difficult problem for us was to find the rank two, graded reflexive module M_2 corresponding to an indecomposable bundle, so-called F_2, which plays a key role in Atiyah's classification (see Theorem 2.6). The theory says that F_2 is given by a short exact sequence of bundles

$$0 \to \mathcal{O}_X \to F_2 \to \mathcal{O}_X \to 0$$

and so we had to study the so-called graded Bourbaki sequences of [6], which we present shortly in Theorem 2.1. We found that M_2 is given modulo shifting by the graded Bourbaki sequence

$$0 \to R(-6) \to M \to \mathfrak{m} \to 0,$$

where $\mathfrak{m} = (Y_1, Y_2, Y_3)R$. In [5] this sequence is called fundamental sequence and M fundamental module. It turned out that M_2 is also the reflexive hull of the differential module of R. For the indecomposability of M (see Lemma 2.5) we noticed that M is the second syzygy of an indecomposable maximal Cohen–Macaulay $R/(Y_3)$-module over R and it was enough to apply a result of [7]. Section 1 presents shortly the main ideas of Atiyah's classification.

1. Preliminaries on Atiyah's classification of vector bundles over elliptic curves

Throughout this section, $(X, \mathcal{O}_X)$ will represent an elliptic curve, which is supposed to be projective, irreducible and smooth. $E \to X$ will be a vector bundle over X and we shall identify it with the locally free sheaf of sections denoted by $\mathcal{E}$. The base field will be denoted by k and it is supposed to be algebraically closed. Let $\mathcal{E}(X)$ be the category of vector bundles. Here we have two operations—the direct sum $\oplus$ and the tensor product $\otimes$. This category satisfies the Krull–Schmidt theorem with respect to $\oplus$, this means that each vector bundle can be decomposed uniquely in indecomposable bundles. We shall study only the indecomposable bundles.

Atiyah gave, in [1], the complete classification of indecomposable vector bundles over elliptic curves. This classification depends on three parameters, two discrete parameters—the rank and the degree of a bundle, and a continuous one—the points of the curve X (such a classification is called "tame" in terms of representation theory).

A *line bundle* is a vector bundle of rank one and for a vector bundle E of rank r we associate a line bundle $\det(E) := \bigwedge^r E$. The *degree of a line bundle* is defined via the well-known isomorphism $\mathrm{Div}(X)/\sim \,\cong \mathrm{Pic}(X)$ (see [8, (II, 6.11), (II, 6.15)]), where $\mathrm{Pic}(X)$ is the set of isomorphism classes of line bundles, $\mathrm{Div}(X)$ is the set of Weil divisors and "$\sim$" denotes the linear equivalence of the divisors. The *degree of a vector bundle* E will be the degree of the line bundle $\det(E)$. Let $P_0 \in X$ be a point. Then the map $P \to \mathcal{O}_X(P - P_0)$ defines a bijection between the points of X and $\mathrm{Pic}^0(X)$—the set of isomorphism classes of line bundles of degree 0 (see [8, (IV, 1.3.7)]).

Let $\mathcal{E}(r,d)$ be the set of isomorphism classes of indecomposable vector bundles of rank r and degree d. So, above we just saw that $\mathcal{E}(1,0)$ is in bijection with the points of X if we fix P_0.

Theorem 1.1 (Atiyah). *There exists a canonical bijection* $\alpha_{r,d} : \mathcal{E}(1,0) \to \mathcal{E}(r,d)$ *for every* $d \in \mathbb{Z}$, $r \in \mathbb{N}^*$.

The construction of $\alpha_{r,d}$ is given by induction, using the following lemmas.

Lemma 1.2. *There exists in* $\mathcal{E}(r,0)$ *a unique up-to isomorphism bundle* F_r *having non-trivial global sections. There exists an exact sequence*

$$0 \to \mathcal{O}_X \to F_r \to F_{r-1} \to 0$$

for all $r \geqslant 2$ *and* $F_1 = \mathcal{O}_X$.

Lemma 1.3. F_r *is self-dual* (*that is,* $F_r^* := \mathrm{Hom}(F_r, \mathcal{O}_X) \cong F_r$), $\det F_r = \mathcal{O}_X$ *and the map* $\alpha_{r,0} : \mathcal{E}(1,0) \to \mathcal{E}(r,0)$ *given by* $L \rightsquigarrow L \otimes F_r$ *is bijective. If* $\mathrm{char}\, k = 0$ *then* $F_r \cong S^{r-1} F_2$ *for all* $r \geqslant 1$, *where* $S^i E$ *denotes the i-symmetric tensor power of* E.

Lemma 1.4. *Let* L *be a line bundle of degree* 1. *Then* $L^* \in \mathcal{E}(1,-1)$ *and the map* $\beta_{r,d} : \mathcal{E}(r,d) \to \mathcal{E}(r, r+d)$ *given by* $E \rightsquigarrow E \otimes L$ *is bijective and its inverse is given by* $G \rightsquigarrow G \otimes L^*$.

Lemma 1.5. *Let $E \in \mathcal{E}(r,d)$, $d \geqslant 1$. Then there exists a unique (up-to isomorphism) bundle $G \in \mathcal{E}(r+d,d)$ given by an extension*

$$0 \to \mathcal{O}_X^d \to G \to E \to 0$$

and the correspondence $E \to G$ defines a bijection $\gamma_{r,d} : \mathcal{E}(r,d) \to \mathcal{E}(r+d,d)$.

Lemma 1.6. *Let $E \in \mathcal{E}(r,d)$. Then $\mathcal{E}(r,d) = \{E \otimes L \mid L \in \mathcal{E}(1,0)\}$. $E \otimes L \cong E$ for some $L \in \mathcal{E}(1,0)$ if and only if $L^{r/\gcd(r,d)} = \mathcal{O}_X$.*

We shall also use the following formulae for the degree of a vector bundle.

Lemma 1.7. *Let $0 \to \Omega(E) \to \bigoplus_{i=1}^s \mathcal{O}_X(n_i) \to E \to 0$ be an exact sequence of bundles, then*

$$\deg(E) + \deg\bigl(\Omega(E)\bigr) = 3\sum_{i=1}^s n_i.$$

Sketch proof of Theorem 1.1. Let for r, d given $h = (r,d)$ be the greatest common divisor of r, d. By Lemma 1.3 it is enough to find a bijection $\mathcal{E}(h,0) \to \mathcal{E}(r,d)$. If $d \geqslant r$, then, using Lemma 1.4 several times, we may reduce to the case when $d < r$. Now, by Lemma 1.5, we may reduce $r := r - d$ and find a bijection $\mathcal{E}(h,0) \to \mathcal{E}(r-d,d)$; proceeding in the same way we can, again, suppose $d \geqslant r$. Clearly, Euclid algorithm shows that these reductions end when $r = h$ and $d = 0$. $\square$

2. Graded maximal Cohen–Macaulay modules over homogeneous cubic hypersurfaces

Let k be an algebraically closed field of characteristic $\neq 3$, f an irreducible homogeneous polynomial of degree three in Y_1, Y_2, Y_3 and $R = k[Y_1, Y_2, Y_3]/(f)$. We shall assume that f is an *isolated singularity*, that is, $R_\mathfrak{p}$ is a regular local ring for all height one prime ideal $\mathfrak{p} \in \operatorname{Spec} R$. A graded *maximal Cohen–Macaulay R-module* (in short MCM) is a graded finitely generated module M with $\operatorname{depth} M = \dim R = 2$ (see [9]). In fact, the graded MCM R-modules are exactly the *reflexive* graded finitely generated R-modules; that is, modules M for which the canonical map $M \to M^{**}$ is an isomorphism, where M^* is the dual of M, that is, $M^* = \operatorname{Hom}_R(M,R)$. A natural way to associate a graded MCM module to any graded finitely generated one is taking the bidual.

A pair of n-square matrices φ, ψ over $k[Y_1, Y_2, Y_3]$ satisfying the conditions $\varphi\psi = \psi\varphi = f \cdot 1_n$ is called *matrix factorization of f*. 1_m denotes the unit $m \times m$

matrix. According to [3], each graded MCM R-module M having no free direct summands, has a 2-periodic minimal free resolution

$$\cdots \to R^n \xrightarrow{\varphi} R^n \xrightarrow{\psi} R^n \xrightarrow{\varphi} R^n \to M \to 0$$

given by a *graded reduced* matrix factorization (φ, ψ), that is, the entries of (φ, ψ) are homogeneous of degree $\geqslant 1$. In this case the first syzygy $\Omega^1_R(M)$ of M is given by (ψ, φ) and we have $M \cong \Omega^2_R(M)$, that is, $M \cong \Omega^1_R(\Omega^1_R(M))$. Note also that $\Omega^1_R(M)$ has no free direct summands if M has none. Using again the periodicity, we see that if M is MCM with no free direct summands, then M is indecomposable if and only if $\Omega^1 M$ is also indecomposable.

Let $X = \operatorname{Proj}(R)$. If M is a graded R-module, then we may find a quasi-coherent sheaf $\widetilde{M}$ over X by *sheafification* and, conversely, given a sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules one defines the *graded R-module associated to $\mathcal{F}$* by $\Gamma_*(\mathcal{F}) = \bigoplus_{n\in\mathbb{Z}} \Gamma(X, \mathcal{F}(n))$, where $\mathcal{F}(n) = \mathcal{O}_X(n) \bigotimes_{\mathcal{O}_X} \mathcal{F}$ (see [8, (II, 5.13)]). Then $\mathcal{F} \cong \widetilde{\Gamma_*(\mathcal{F})}$ and the canonical map $\delta_m : M \to \Gamma_*(\widetilde{M})$ is an isomorphism if and only if $\operatorname{depth}_R M \geqslant 2$ by a well-known theorem of Grothendieck and Serre (see [10]). If M is a graded MCM R-module, then δ_M is an isomorphism and $M_\mathfrak{p}$ is a free $R_\mathfrak{p}$-module for all height one prime ideals $\mathfrak{p}$ of R (MCM modules over regular local rings are free!). This results in an equivalence between graded MCM R-modules and $\mathcal{E}(X)$ given by $M \to \widetilde{M}$. We need a small dictionary to translate the terms from bundle theory into the terms of MCM modules: for instance, when does an inclusion $N \subset M$ of graded MCM R-modules define a bundle inclusion $\widetilde{N} \subset \widetilde{M}$? Note that $\widetilde{N} \subset \widetilde{M}$ is a bundle inclusion if on level of points $\mathfrak{p}$; that is, graded $\mathfrak{p} \in \operatorname{Spec} R$, $\operatorname{ht} \mathfrak{p} = 1$, we have a retraction of the inclusion between free modules, $N_\mathfrak{p} \subset M_\mathfrak{p}$. In particular, this means that $(M/N)_\mathfrak{p}$ is torsion-free for all graded height one prime ideals $\mathfrak{p}$ of R. Thus, $N \subset M$ induces a bundle inclusion if the annihilator ideal of any non-zero element of M/N is an $\mathfrak{m}$-primary ideal ($\mathfrak{m}$ denotes $(Y_1, Y_2, Y_3)R$) or 0, the first possibility fails because $\operatorname{depth}_R M/N \geqslant 1$ by Depth Lemma. Hence, $N \subset M$ induces a bundle inclusion if and only if M/N is torsion-free, the sufficiency being easy, since finitely generated torsion-free modules over a *DVR* are free.

A short exact sequence of graded R-modules

$$0 \to N \to M \to M/N \to 0$$

with N, M graded MCM R-modules induces an extension of bundles

$$0 \to \widetilde{N} \to \widetilde{M} \to \mathcal{E} \to 0$$

if and only if M/N is torsion-free and $\mathcal{E}$ is the sheafification of M/N or, equivalently, $\mathcal{E} \cong ((M/N)^{**})^{\sim}$. If $\mathcal{E} = \mathcal{O}_X$ then $(M/N)^{**} \cong R$ and M/N can be identified with a graded ideal I of R which must be $\mathfrak{m}$-primary because $I^{**} \cong R$. According to [6] a *graded Bourbaki sequence* is an exact sequence of graded R-modules

$$0 \to F \to M \to I \to 0,$$

where F is free, M is a graded MCM R-module and I is a graded $\mathfrak{m}$-primary ideal, or $I = R$ (see also Theorem 6 of [11, §4, no. 9]).

An R-module M of rank r is *orientable* if $(\bigwedge^r M)^{**} \cong R$. So the graded orientable MCM R-modules induce bundles of degree multiple of 3 ($\deg R(1) = 3$!) and the MCM R-module M_2 corresponding to F_2 (see Lemma 1.3) is orientable. But which is really M_2? In the next part of this section we shall solve this question for the case $f = Y_1^3 + Y_2^3 + Y_3^3$ (R is an isolated singularity because $\operatorname{char} k \neq 3$).

By Serre's condition, we see that R is normal (R is Cohen–Macaulay and an isolated singularity). Using graded variants of Propositions 1.8, 1.9, Lemmas 1.10, 1.11, and Theorem 3.1(b) of [6], we obtain the following theorem.

Theorem 2.1 (Herzog–Kühl).

(1) *If M is a graded orientable R-module, then there exists a graded Bourbaki sequence*

$$0 \to F \to M \to I \to 0.$$

(2) *If I is a graded $\mathfrak{m}$-primary ideal and $s = \dim_k \operatorname{Soc}(R/I)$ then there exists a graded Bourbaki sequence*

$$0 \to F_s \to M \to I \to 0$$

with F_s free of rank s and such M is graded, orientable and uniquely (up-to isomorphism) determined by I. Moreover, M is a direct sum of $\Omega_R^2(I)$ and a free module. If $\Omega_R^1(I)$ has no free direct summands, then $\mu(M) = s + \mu(I)$, where $\mu(M)$ denotes the minimal number of generators of M.

(3) *Let I be an $\mathfrak{m}$-primary ideal. Then an extension of graded R-modules*

$$(\xi)\ 0 \to R(-t) \to N \to I \to 0$$

is a graded Bourbaki sequence (that is, N is MCM) if and only if $\operatorname{Ext}_R^1(I, R) \cong \omega_{R/I}$ (the canonical module ω of R/I) is a cyclic R-module (in particular, R/I is Gorenstein) and (ξ) is a generator of $\operatorname{Ext}_R^1(I, R)$ [9].

(4) *There exist non-free orientable graded MCM R-modules M of rank* 2 *only with $\mu(M) = 4$, or* 6.

For the proof we mention only that (3) follows from the proof of Proposition 1.9 of [6].

Remark 2.2. Lemmas 1.2 and 1.3 say that F_2 is self-dual, $F_2 \wedge F_2 \cong \mathcal{O}_X$ and we have an exact sequence

$$0 \to \mathcal{O}_X \to F_2 \to \mathcal{O}_X \to 0.$$

So the graded MCM R-module M_2 corresponding to F_2 is orientable with $M_2^* \cong M_2$ and such that there exists a Bourbaki sequence

$$0 \to R(-t) \to M_2 \to I \to 0, \quad t \in \mathbb{N},$$

with $\dim_k \operatorname{Soc}(R/I) = 1$, that is, R/I is Gorenstein. By the above theorem, $\mu(M_2)$ can be 4, or 6 and if $\Omega_R^1(I)$ has no free direct summands then $\mu(I)$ can be 3, or 5. Note that $\mu(\mathfrak{m}) = 3$ and $k = R/\mathfrak{m}$ is Gorenstein Artinian.

Lemma 2.3. *Let*

$$\rho = \begin{pmatrix} Y_1^2 & -Y_2 & -Y_3 & 0 \\ Y_2^2 & Y_1 & 0 & -Y_3 \\ Y_3^2 & 0 & Y_1 & Y_2 \end{pmatrix}, \qquad \psi = \begin{pmatrix} Y_1 & Y_2 & Y_3 & 0 \\ -Y_2^2 & Y_1^2 & 0 & Y_3 \\ -Y_3^2 & 0 & Y_1^2 & -Y_2 \\ 0 & -Y_3^2 & Y_2^2 & Y_1 \end{pmatrix}$$

and φ the square 4*-matrix obtained from ρ adding a fourth row*

$$\gamma = \begin{pmatrix} 0 & Y_3^2 & -Y_2^2 & Y_1^2 \end{pmatrix},$$

that is, $\varphi = \binom{\rho}{\gamma}$. Then (ψ, φ) is a matrix factorization of $\Omega_R^1(\mathfrak{m})$ and the following exact sequence:

$$\xrightarrow{\psi} R(-3) \oplus R(-2)^3 \xrightarrow{\rho} R(-1)^3 \xrightarrow{(Y_1, Y_2, Y_3)} \mathfrak{m} \to 0$$

is a minimal free graded resolution of $\mathfrak{m}$. In particular, $\Omega_R^1(\mathfrak{m})$ has no free direct summands.

Proof. It is easy to see that $\varphi\psi = f \cdot 1_4$, $f = Y_1^3 + Y_2^3 + Y_3^3$. Then $\psi\varphi = f \cdot 1_4$ and the above sequence is a complex since ρ is part of φ and $(Y_1, Y_2, Y_3, 0)$ is the first line of ψ. Let $u_1, u_2, u_3 \in R$ be such that $\sum_{i=1}^3 Y_i u_i = 0$. We show that $u = (u_1\ u_2\ u_3)^{\mathrm{T}}$ belongs to $\operatorname{Im}\rho$, that is, to the module generated by the columns of ρ. Subtracting multiples of the columns 2 and 3 of ρ from u, we may suppose that u_1 depends only on Y_1. As the maps are graded, we may suppose u is graded and so u_1 has the form εY_1^s, where $\varepsilon \in k$ and $s \in \mathbb{N}$. If $\varepsilon \neq 0$, then the equation $\sum_{i=1}^3 Y_i u_i = 0$ in R gives necessarily $s + 1 \equiv 0 \bmod 3$ and subtracting from u multiples of the first column of ρ we reduce to the case $u_1 = 0$. Then $Y_2 u_2 + Y_3 u_3 = 0$ and since $\{Y_2, Y_3\}$ is a regular sequence in R we see that u is a multiple of the column 4 of ρ.

Next we show that $\operatorname{Ker}\rho \subset \operatorname{Im}\psi$. Since $\operatorname{Im}\psi = \operatorname{Ker}\varphi$ ((φ, ψ) is a matrix factorization!) it is enough to show that $\operatorname{Ker}\rho \subset \operatorname{Ker}\varphi$. Let δ_2, δ_3 the rows 2, 3 of ρ. We have $Y_2^2\delta_3 - Y_3^2\delta_2 + Y_1\gamma = 0$. If $\nu \in \operatorname{Ker}\rho$, then $\delta_3\nu = \delta_2\nu = 0$ and we obtain $Y_1\gamma\nu = 0$. But Y_1 is non-zero divisor in R and so $\gamma\nu = 0$, which is enough since $\varphi = \binom{\rho}{\gamma}$.

Hence, the sequence is exact and so (ψ, φ) is a matrix factorization of $\Omega_R^1(\mathfrak{m})$. Thus, $\Omega_R^1(\mathfrak{m})$ has no free direct summands. $\square$

Proposition 2.4. *There exists a graded Bourbaki exact sequence*

$$0 \to R(-6) \to \Omega_R^2(\mathfrak{m}) \to \mathfrak{m} \to 0$$

and $R(3) \otimes \Omega_R^2(\mathfrak{m})$ corresponds to a bundle of degree 0 and rank 2, which is self-dual.[1]

Proof. By Lemma 2.3 we have the following exact sequence

$$0 \to \Omega_R^2(\mathfrak{m}) \to R(-3) \oplus R(-2)^3 \xrightarrow{\rho} R(-1)^3 \xrightarrow{(Y_1,Y_2,Y_3)} \mathfrak{m} \to 0.$$

Thus $\Omega_R^2(\mathfrak{m})$ corresponds to a bundle of degree

$$\deg\bigl(R(-3) \oplus R(-2)^3\bigr) - \deg\bigl(R(-1)^3\bigr) = -27 + 9 = -18 = \deg\bigl(R(-6)\bigr).$$

So $R(3) \otimes \Omega_R^2(\mathfrak{m})$ corresponds to a bundle of degree 0.

By Theorem 2.1(2), we have a graded Bourbaki sequence

$$0 \to R(-t) \to M \to \mathfrak{m} \to 0, \quad t \in \mathbb{N},$$

and M is a direct sum of $\Omega_R^2(\mathfrak{m})$ and a free module. But $\Omega_R^1(\mathfrak{m})$ has no free direct summands by Lemma 2.3 and so $\mu(M) = 1 + \mu(m) = 4$. As $\mu(\Omega_R^2(\mathfrak{m})) = 4$ we obtain $M \cong \Omega_R^2(\mathfrak{m})$. The above graded Bourbaki sequence gives that M corresponds to a bundle of degree exactly $\deg R(-t)$, so t must be 6. Dualizing the above graded Bourbaki sequence, we obtain the same sequence after some shifting because $\mathrm{Ext}_R^1(\mathfrak{m}, R) = k$. $\square$

Lemma 2.5. *Let P be the MCM R-module given over $\overline{R} := R/(Y_3) \cong k[Y_1, Y_2]/(Y_1^3 + Y_2^3)$ by the matrix factorizations (θ, δ),*

$$\theta = \begin{pmatrix} Y_1^2 & -Y_2 \\ Y_2^2 & Y_1 \end{pmatrix}, \qquad \delta = \begin{pmatrix} Y_1 & Y_2 \\ -Y_2^2 & Y_1^2 \end{pmatrix}.$$

Then

(1) $\Omega_R^1(P) \cong \Omega_R^1(\mathfrak{m})$;
(2) $\Omega_R^1(\mathfrak{m}), \Omega_R^2(\mathfrak{m})$ *are indecomposable modules.*

Proof. Let

$$\tau = (\theta \mid -Y_3 1_2) \quad \text{and} \quad \eta = \left(\begin{array}{c|c} \delta & Y_3 1_2 \\ \hline -Y_3^2 1_2 & \theta \end{array}\right).$$

[1] Note that $\Omega_R^2(\mathfrak{m})$ is the reflexive hull of the module of differentials of R (cf. Section 5). Yoshino [5] calls this sequence fundamental sequence and $\Omega_R^2(\mathfrak{m})$ fundamental module. $\Omega_R^1(\mathfrak{m})$ and $\Omega_R^2(\mathfrak{m})$ can also be obtained using the construction described in [12]. They associate to $f = \sum_{i=1}^3 w_i y_i$ a matrix factorization $M(w, y)$. Applying this to $w_i = x_i$ and $y_i = x_i^2$ we obtain the modules above.

The sequence

$$R^4 \xrightarrow{\eta} R^4 \xrightarrow{\tau} R^2 \to P \to 0$$

is exact. Indeed $\operatorname{Coker}\tau = \overline{R}^2/\operatorname{Im}\theta \cong P$ and $\tau\eta = 0$, because τ is given by the first rows of φ and $\eta = \psi$, where φ, ψ were defined in Lemma 2.3. If $u, v \in R^2$ satisfies $\tau\binom{u}{v} = 0$, then $\theta u - Y_3 v = 0$. Subtracting from $\binom{u}{v}$ some multiples of columns 3, 4 of η (these are in $\operatorname{Im}\eta$!), we may suppose that u does not depend on Y_3. Then $\theta u \equiv 0 \bmod Y_3$ implies $u = \delta w$ for a $w \in R^2$. So, subtracting from $\binom{u}{v}$ some multiples of the first two columns of η, we may reduce to $u = 0$. Then $Y_3 v = 0$ and so $v = 0$ because Y_3 is non-zero divisor in R.

The above sequence says that $\Omega^1_R(P) \cong \operatorname{Im}\eta = \operatorname{Im}\psi \cong \Omega^1_R(\mathfrak{m})$. Since P is an indecomposable MCM $\overline{R}$-module (see, for example, the list of indecomposable MCM modules over singularity D_4 [5]), we see by Theorem 4.1 of [7] that $\Omega^1_R(P)$ is indecomposable, too. Then $\Omega^1_R(\mathfrak{m})$ is indecomposable and so $\Omega^2_R(\mathfrak{m})$ is, too.[2] $\square$

Theorem 2.6. *$M_2 = R(3) \otimes \Omega^2_R(\mathfrak{m})$ is the unique (up-to isomorphism) indecomposable graded, orientable, self-dual MCM R-module of degree 0 and rank 2. It corresponds to the bundle $F_2 \in \mathcal{E}(2, 0)$.*

Proof. The first sentence follows from Proposition 2.4 and Lemma 2.5, except the uniqueness. Thus M_2 corresponds to a bundle $\mathcal{E}$ from $\mathcal{E}(2, 0)$ with non-trivial global sections (it contains $\mathcal{O}$!). By Lemma 1.2, $\mathcal{E}$ is unique with such property and $\mathcal{E} = F_2$. $\square$

3. Rank one maximal Cohen–Macaulay modules over $Y_1^3 + Y_2^3 + Y_3^3$

As usual, let $R = k[Y_1, Y_2, Y_3]/(f)$, $f = Y_1^3 + Y_2^3 + Y_3^3$, k being an algebraically closed field with $\operatorname{char} k \neq 3$. If $\lambda = [\lambda_1 : \lambda_2 : 1]$ is a point of $V(f) \subset \mathbb{P}^2_k$, denote

$$\varphi_\lambda = \begin{pmatrix} Y_1 - \lambda_1 Y_3 & -\left(Y_2^2 + \lambda_2 Y_2 Y_3 + \lambda_2^2 Y_3^2\right) \\ Y_2 - \lambda_2 Y_3 & Y_1^2 + \lambda_1 Y_1 Y_3 + \lambda_1^2 Y_3^2 \end{pmatrix},$$

$$\psi_\lambda = \begin{pmatrix} Y_1^2 + \lambda_1 Y_1 Y_3 + \lambda_1^2 Y_3^2 & Y_2^2 + \lambda_2 Y_2 Y_3 + \lambda_2^2 Y_3^2 \\ -(Y_2 - \lambda_2 Y_3) & Y_1 - \lambda_1 Y_3 \end{pmatrix}.$$

If $\lambda = [\lambda_1 : 1 : 0] \in V(f)$ we set

[2] This result was also proved by Kawamoto und Yoshino (cf. [13]). They proved that M_2 is decomposable for a normal local two-dimensional domain R only if R is a cyclic quotient singularity.

$$\varphi_\lambda = \begin{pmatrix} Y_1 - \lambda_1 Y_2 & -Y_3^2 \\ Y_3 & Y_1^2 + \lambda_1 Y_1 Y_2 + \lambda_1^2 Y_2^2 \end{pmatrix},$$

$$\psi_\lambda = \begin{pmatrix} Y_1^2 + \lambda_1 Y_1 Y_2 + \lambda_1^2 Y_2^2 & Y_3^2 \\ -Y_3 & Y_1 - \lambda_1 Y_2 \end{pmatrix}.$$

Theorem 3.1. $(\varphi_\lambda, \psi_\lambda)$ *is a matrix factorization for all* $\lambda \in V(f)$ *and the sets of graded MCM modules* $\mathcal{M}_{-1} = \{\operatorname{Coker}\varphi_\lambda \mid \lambda \in V(f)\}$, $\mathcal{M}_1 = \{\operatorname{Coker}\psi_\lambda \mid \lambda \in V(f)\}$ *have the following properties*:

(1) *Every two-generated non-free graded MCM R-module is isomorphic with one of the modules from* $\mathcal{M}_1 \cup \mathcal{M}_{-1}$.[3]
(2) *Every two different graded MCM R-modules from $\mathcal{M}_1 \cup \mathcal{M}_{-1}$ are not isomorphic.*
(3) *The modules from $\mathcal{M}_1$ are the syzygies and also the duals of the modules from $\mathcal{M}_{-1}$.*
(4) *The modules from $\mathcal{M}_1, \mathcal{M}_{-1}$ all have rank one.*

Proof. Clearly $\varphi_\lambda\psi_\lambda = \psi_\lambda\varphi_\lambda = f \cdot 1_2$. It is easy to see that no elementary transformations can transform φ_λ into a $\psi_{\lambda'}$ for $\lambda, \lambda' \in V(f)$. Indeed, let U be an invertible 2×2 matrix over $k[X]$ and U' the homogeneous part of U of degree 0, i.e., the entries of U' are the constant terms of the entries of U. Then U' is still invertible if $V = U^{-1}$, then $V' = (U')^{-1}$. If $U\varphi_\lambda = \psi_{\lambda'}V$ for some points $\lambda, \lambda' \in V(f)$, then it follows that $U'\varphi_\lambda = \psi_{\lambda'}V$. Note that the degree of entry $(1,1)$ in $U'\varphi_\lambda$ is 1 but the degree of the entry $(1,1)$ in $\psi_{\lambda'}V'$ is 2. It follows that the entry $(1,1)$ in $U'\varphi_\lambda$ is 0. Then the first row of U' must be zero because the entries of the first column of φ_λ are linearly independent over k. But this is not possible since U' is invertible. Thus, no MCM R-module of $\mathcal{M}_1$ is isomorphic with one of $\mathcal{M}_{-1}$. The rest of (1), (2) is proved in Proposition 1.1 of [14], where, by mistake, we forget about $\mathcal{M}_{-1}$ but this could be done similarly. By construction, the modules of $\mathcal{M}_1$ are the syzygies of the modules of $\mathcal{M}_{-1}$. Since the transpose of φ_λ is exactly $\binom{0\ 1}{1\ 0}\psi_\lambda\binom{0\ 1}{1\ 0}$ we see that $\operatorname{Coker}\psi_\lambda$ is isomorphic with the dual of $\operatorname{Coker}\varphi_\lambda$.

Now, clearly, $\operatorname{rank}\operatorname{Coker}\varphi_\lambda + \operatorname{rank}\operatorname{Coker}\psi_\lambda = 2$ because we have an exact sequence of graded modules

$$0 \to \operatorname{Coker}\psi_\lambda \to F \to \operatorname{Coker}\varphi_\lambda \to 0,$$

where F is free of rank 2. So (4) holds, too. $\square$

Next we describe all three-generated, rank one, graded MCM R-modules. By Corollary 1.3 of [6], we have $\mu(M) \leqslant 3 \operatorname{rank} M$ for all graded MCM R-

[3] Note that according to our aim as isomorphism of graded modules are also allowed isomorphisms of degree different from zero.

modules M. Thus, the graded MCM R-modules of rank 1 are generated by, at most, three elements. The following lemma gives mainly the form of three-generated, rank one, graded MCM R-modules.

Lemma 3.2. *Let ρ_1, ρ_2, w_1, w_2 be linear forms of $k[Y_1, Y_2, Y_3]$ such that*

(1) *f is contained in the intersections of the ideals (ρ_1, ρ_2), (w_1, w_2) of $k[Y_1, Y_2, Y_3]$;*
(2) *$\{\rho_1, w_1, w_2\}$, $\{\rho_2, w_1, w_2\}$ are linearly k-independent systems of linear forms in $k[Y_1, Y_2, Y_3]$.*

Then there exist linear forms a, b, c, d such that

$$\det\begin{pmatrix} 0 & \rho_1 & \rho_2 \\ w_1 & a & b \\ w_2 & c & d \end{pmatrix} = f.$$

Proof. By (1) there exist two degree forms η_1, η_2 such that $f = \eta_1\rho_1 + \eta_2\rho_2$, which are not unique. Note that $\eta_1' = \eta_1 + \rho_2\delta$, $\eta_2' = \eta_2 - \rho_1\delta$ satisfy also $f = \eta_1'\rho_1 + \eta_2'\rho_2$ for any linear form δ. We show that for some δ there exist a, b, c, d linear forms such that

$$\begin{vmatrix} w_1 & b \\ w_2 & d \end{vmatrix} = -\eta_1' \quad \text{and} \quad \begin{vmatrix} w_1 & a \\ w_2 & c \end{vmatrix} = \eta_2'.$$

By (2) we have

(a) $\eta_1 = \rho_2\eta_{11} + w_1\eta_{12} + w_2\eta_{13}$;
(b) $\eta_2 = \rho_1\eta_{21} + w_1\eta_{22} + w_2\eta_{23}$

for some linear forms η_{ij}. If $\eta_{11} = \eta_{21} = 0$ then we may take $a = -\eta_{23}$, $c = \eta_{22}$, $b = -\eta_{13}$, $d = \eta_{12}$ and $\delta = 0$ above. If not, let us say $\eta_{11} \neq 0$, then we may suppose even $\eta_{11} \notin \langle w_1, w_2\rangle$ because, otherwise, we may reduce to the case $\eta_{11} = 0$.

Then $\{\eta_{11}, w_1, w_2\}$ is a linearly k-independent system of linear forms and we may express

$$\eta_{21} = \lambda_1\eta_{11} + \lambda_2 w_1 + \lambda_3 w_2 \quad \text{for } \lambda_1, \lambda_2, \lambda_3 \in k.$$

Substituting η_{21} in (b) we obtain

(b') $\eta_2' = \eta_2 - \lambda_1\rho_1\eta_{11} = \begin{vmatrix} w_1 & a \\ w_2 & c \end{vmatrix}$.

for some linear forms a, c. Also note that (a) says that there exist b, d linear forms such that

$$\eta_1' = \eta_1 - \rho_2\eta_{11} \quad \text{satisfies } -\eta_1' = \begin{vmatrix} w_1 & b \\ w_2 & d \end{vmatrix}.$$

To finish we must see that $\lambda_1 = -1$ because, then, $\delta = -\eta_{11}$ works.

We have $f = \eta_1\rho_1 + \eta_2\rho_2 \equiv \rho_1\rho_2\eta_{11}(1+\lambda_1)$ modulo $\langle \eta_1', \eta_2' \rangle$. As $\langle \eta_1', \eta_2' \rangle \subset \langle w_1, w_2 \rangle$ we obtain $(\lambda_1+1)\rho_1\rho_2\eta_{11} \in (w_1, w_2)$ because $f \in (w_1, w_2)$ by (1). But (w_1, w_2) is a prime ideal since $\{w_1, w_2\}$ are linearly k-independent linear forms. By (2) and by choice of η_{11} we have $\rho_1, \rho_2, \eta_{11} \notin (w_1, w_2)$. Then $(\lambda_1+1)\rho_1\rho_2\eta_{11}$ is not contained in (w_1, w_2) unless $\lambda_1 = -1$. □

Lemma 3.3. *Let φ_1, φ_2 be linear forms in $k[Y_1, Y_2, Y_3]$. Then there exist no linear forms a, b, c, d such that*

$$\det\begin{pmatrix} 0 & \varphi_1 & \varphi_2 \\ \varphi_1 & a & b \\ \varphi_2 & c & d \end{pmatrix} = f.$$

Proof. If $\{\varphi_1, \varphi_2\}$ is linearly dependent, then there are no such a, b, c, d because f is irreducible. Suppose now $\{\varphi_1, \varphi_2\}$ is linearly independent and there exist a, b, c, d as above. Then $f = -\varphi_1^2 d - \varphi_2^2 a + \varphi_1\varphi_2(c+b) \in (\varphi_1^2, \varphi_1\varphi_2, \varphi_2^2)$. Let τ be a linear transformation sending (φ_1, φ_2) in (Y_1, Y_2), let us say $\tau(Y_i) = u_i$. Thus, $\{u_1, u_2, u_3\}$ are linearly independent linear forms and

$$\tau(f) = u_1^3 + u_2^3 + u_3^3 \in \left(Y_1^2, Y_1Y_2, Y_2^2\right).$$

Put $Y_1 = Y_2 = 0$ in $\tau(f)$ and we obtain $\sum_{i=1}^{3} \bar{u}_i^3 = 0$ for $\bar{u}_i := u_i(Y_1 = Y_2 = 0)$. This is a contradiction since u_1, u_2, u_3 are linearly independent. □

If $\lambda = [\lambda_1 : \lambda_2 : 1]$ is a point of $V(f) \in \mathbb{P}_k^2$, set $\rho_{1\lambda} = Y_1 - \lambda_1 Y_3$, $\rho_{2\lambda} = Y_2 - \lambda_2 Y_3$. If $\lambda = [\lambda_1 : 1 : 0] \in V(f)$, set $\rho_{1\lambda} = Y_1 - \lambda_1 Y_2$, $\rho_{2\lambda} = Y_3$.

Proposition 3.4. *Let M be a three-generated, rank one, graded MCM R-module. Then there exist $\lambda, \xi \in V(f)$, $\lambda \neq \xi$ and some linear forms a, b, c, d such that*

$$\varphi = \begin{pmatrix} 0 & \rho_{1\lambda} & \rho_{2\lambda} \\ \rho_{1\xi} & a & b \\ \rho_{2\xi} & c & d \end{pmatrix}$$

and its adjoint matrix ψ form a matrix factorization of M.

Proof. As rank $M = 1$ every matrix factorization (φ', ψ') of M has $\det \varphi' = f$ (see [3, (6.4)]). It is enough to show that φ' has a generalized zero, that is, φ' receives an entry zero (we may suppose in the position $(1,1)$) after some elementary transformations. Indeed, then φ' could be arranged in the required form by some elementary transformations on lines 2, 3 and columns 2, 3. We have $\lambda \neq \xi$ because of Lemma 3.3. But we obtain a generalized zero for φ' by applying the following theorem.

Theorem 3.5 (Eisenbud [15]). *Suppose $a = (g_{ij})_{i,j}$ is an $n \times n$ matrix of linear forms with no generalized zeros. Then* $\det a \not\equiv 0$ *mod* $(h_1, \dots, h_{n-1})$ *for any linear forms* $h_1, \dots, h_{n-1}$.

Back to our proof, we see that if φ' would not have any generalized zeros, then $f = \det \varphi' \notin (h_1, h_2)$ for any linear forms. But $f \in (\rho_{1\lambda}, \rho_{2\lambda})$ for any $\lambda \in V(f)$. Contradiction!

An elementary different proof can easily be obtained by subtracting from the first column of φ' the second column multiplied with $\alpha \in k$ and the third column multiplied with $\beta \in k$ (α, β to be determined). Then, in the new column, write that the third entry is a multiple of the first one with γ. This condition gives three equations on α, β, γ, identifying the coefficients of Y_1, Y_2, Y_3 which finally means a cubic monic equation in α (k is algebraically closed!). □

Proposition 3.6. *Let M be a three-generated, rank one, graded MCM R-module and $\lambda, \xi \in V(f)$, $\lambda \neq \xi$, $a, b, c, d, \varphi, \psi$ as in Proposition* 3.4. *Then, for each $\eta \in V(f)$, there exists $\theta \in V(f)$, $\theta \neq \eta$ and some linear forms a', b', c', d' such that*

$$\varphi' = \begin{pmatrix} 0 & \rho_{1\eta} & \rho_{2\eta} \\ \rho_{1\theta} & a' & b' \\ \rho_{2\theta} & c' & d' \end{pmatrix}$$

and its adjoint matrix ψ' form another matrix factorization (φ', ψ') of M.

Proof. Let $U = (u_{ij})$, $V = (v_{ij})$ be invertible 3×3 matrix over k, where u_{ij}, v_{ij} are parameters. After a renumeration of Y, we may suppose $\lambda_3 = \xi_3 = 1$. We shall give here the proof only in the case $\eta_3 = 1$, the case $\eta_3 = 0$ being similar. We want to find U, V such that the first line in $\varphi' = U\varphi V^{-1}$ is $(0\ \rho_{1\eta}\ \rho_{2\eta})$, that is, the first line in $U\varphi$ should be $(0\ \rho_{1\eta}\ \rho_{2\eta})V$. Identifying the entries we obtain

$$u_{12}\rho_{1\xi} + u_{13}\rho_{2\xi} = \rho_{1\eta}v_{21} + \rho_{2\eta}v_{31},$$
$$u_{11}\rho_{1\lambda} + u_{12}a + u_{13}c = \rho_{1\eta}v_{22} + \rho_{2\eta}v_{32},$$
$$u_{11}\rho_{2\lambda} + u_{12}b + u_{13}d = \rho_{1\eta}v_{23} + \rho_{2\eta}v_{33}.$$

We should see that we are able to find U, V invertibly satisfying the above system. Identifying the coefficients of Y_i in the system we obtain 9 equations:

$$u_{12} = v_{21}, \qquad u_{13} = v_{31}, \qquad -u_{12}\xi_1 - u_{13}\xi_2 = -\eta_1 v_{21} - \eta_2 v_{31},$$
$$u_{11} + u_{12}a_1 + u_{13}c_1 = v_{22}, \qquad u_{12}a_2 + u_{13}c_2 = v_{32},$$
$$-\lambda_1 u_{11} + u_{12}a_3 + u_{13}c_3 = -\eta_1 v_{22} - \eta_2 v_{32},$$
$$u_{12}b_1 + u_{13}d_1 = v_{23}, \qquad u_{11} + u_{12}b_2 + u_{13}d_2 = v_{33},$$
$$-u_{11}\lambda_2 + u_{12}b_3 + u_{13}d_3 = -\eta_1 v_{23} - \eta_2 v_{33},$$

where $a=\sum a_iY_i$, $b=\sum b_iY_i$, $c=\sum c_iY_i$, $a_i,b_i,c_i\in k$.

Eliminate v_{ij} from the system and we obtain an homogeneous system of three equations in u_{11},u_{12},u_{13} whose coefficient matrix A is exactly the transpose of φ, where we substitute $Y_1=\eta_1$, $Y_2=\eta_2$, $Y_3=1$. Then $\det A=f(\eta)=0$ and so we may choose a non-zero solution u_{11},u_{12},u_{13}, which can be completed to an invertible matrix U and similarly v_{11},v_{21},v_{31} which can be completed to an invertible matrix V. We may take φ' to be $U\varphi V^{-1}$ multiplied with a non-zero element of k. $\square$

Let $P_0=[-1:0:1]\in V(f)$. For each $\lambda=[\lambda_1:\lambda_2:1]\in V(f)$, $\lambda\neq P_0$ set

$$\alpha_\lambda=\begin{pmatrix}0 & \rho_{1\lambda} & \rho_{2\lambda}\\ Y_1+Y_3 & -Y_2-\lambda_2Y_3 & -wY_3\\ Y_2 & wY_3 & (-\lambda_1+1)Y_3-Y_1\end{pmatrix},$$

where $w=\lambda_2^2/(\lambda_1+1)$. (If $\lambda_1=-1$ then we obtain $\lambda_2=0$ since $\lambda\in V(f)$ and so $\lambda=P_0$. Contradiction!) As in Proposition 3.4 we set $\rho_{1\lambda}=Y_1-\lambda_1Y_3$, $\rho_{2\lambda}=Y_2-\lambda_2Y_3$. If $[\lambda_1:1:0]$ set

$$\alpha_\lambda=\begin{pmatrix}0 & \rho_{1\lambda} & \rho_{2\lambda}\\ Y_1+Y_3 & -\lambda_1Y_1 & \lambda_1Y_1+\lambda_1^2Y_2\\ Y_2 & Y_3-Y_1 & -Y_1\end{pmatrix},$$

where $\rho_{1\lambda}=Y_1-\lambda_1Y_2$, $\rho_{2\lambda}=Y_3$ as in Proposition 3.4. Let β_λ be the adjoint matrix of α_λ.

Theorem 3.7. *$(\alpha_\lambda,\beta_\lambda)$ is a matrix factorization for all $\lambda\in V(f)$, $\lambda\neq P_0$ and the set of three-generated MCM graded R-modules $\mathcal{M}_0=\{\operatorname{Coker}\alpha_\lambda\mid\lambda\in V(f),\ \lambda\neq P_0\}$ has the following properties*:

(1) *the modules from $\mathcal{M}_0$ have all ranks one*;
(2) *every two different modules from $\mathcal{M}_0$ are not isomorphic*;
(3) *every three-generated, rank one, non-free, graded MCM R-module is isomorphic with one module from $\mathcal{M}_0$.*

Proof. Note that $\alpha_\lambda\beta_\lambda=\beta_\lambda\alpha_\lambda=f\ 1_3$ because $\det\alpha_\lambda=f$. By [3, (6.4)] we obtain then rank $(\operatorname{Coker}\alpha_\lambda)=1$. For (2) we suppose that there exist two invertible matrices U,V over k of determinant 1 such that $U\alpha_\lambda=\alpha_\xi V$ for $\lambda,\xi\in V(f)\setminus P_0$. Identifying the entries of $U\alpha_\lambda,\alpha_\xi V$ and the coefficients of Y_i we obtain a big system of equations. Using SINGULAR [16], we obtain in Lemma 5.1, with the help of a computer, $\lambda=\xi$.

(3) By Proposition 3.4 given a three-generated, rank one, non-free, graded MCM R-module there exist $\lambda,\xi\in V(f)$, $\lambda\neq\xi$, and some linear forms a,b,c,d such that

$$\varphi=\begin{pmatrix}0 & \rho_{1\lambda} & \rho_{2\lambda}\\ \rho_{1\xi} & a & b\\ \rho_{2\xi} & c & d\end{pmatrix}$$

and its adjoint matrix form a matrix factorization of M. By Proposition 3.6 we may suppose $\xi = P_0$. It is enough to show that, after elementary transformations, φ will become α_λ.

Set the following forms of degree two:

$$\gamma = \begin{vmatrix} \rho_{1\xi} & a \\ \rho_{2\xi} & c \end{vmatrix}, \qquad \delta = \begin{vmatrix} \rho_{1\xi} & b \\ \rho_{2\xi} & d \end{vmatrix} \qquad \text{and}$$
$$\bar\gamma = \begin{vmatrix} \rho_{1\xi} & \bar a \\ \rho_{2\xi} & \bar c \end{vmatrix}, \qquad \bar\delta = \begin{vmatrix} \rho_{1\xi} & \bar b \\ \rho_{2\xi} & \bar d \end{vmatrix},$$

where the linear forms $\bar a, \bar b, \bar c, \bar d$ are given in α_λ. We have $-\rho_{1\lambda}\delta + \rho_{2\lambda}\gamma = \det\varphi = f = -\rho_{1\lambda}\bar\delta + \rho_{2\lambda}\bar\gamma$ and it follows $\rho_{2\lambda}(\gamma - \bar\gamma) = \rho_{1\lambda}(\delta - \bar\delta)$. As $\{\rho_{1\lambda}, \rho_{2\lambda}\}$ is a regular sequence, we obtain $\gamma - \bar\gamma = \rho_{1\lambda}\varepsilon$, $\delta - \bar\delta = \rho_{2\lambda}\varepsilon$ for a linear form ε. By construction $\gamma, \bar\gamma \in (\rho_{1\xi}, \rho_{2\xi})$ and so $\rho_{1\lambda}\varepsilon \in (\rho_{1\xi}, \rho_{2\xi})$. But $\rho_{1\lambda}, \rho_{1\xi}, \rho_{2\xi}$ are three linearly independent linear forms and, therefore, they form a regular sequence in $k[Y_1, Y_2, Y_3]$. It follows $\varepsilon \in (\rho_{1\xi}, \rho_{2\xi})$, let us say $\varepsilon = u_1\rho_{1\xi} + u_2\rho_{2\xi}$ for some $u_1, u_2 \in k$ (degree reason!).

Subtracting the first line of φ multiplied by u_2 from the second line and adding the first line multiplied by u_1 to the third line, we obtain some new γ', δ' such that

$$\gamma' = \gamma - \begin{vmatrix} \rho_{1\xi} & -u_2 \\ \rho_{2\xi} & u_1 \end{vmatrix} \rho_{1\lambda} = \gamma - \varepsilon\rho_{1\lambda} = \bar\gamma \quad \text{and, similarly,} \quad \delta' = \bar\delta.$$

Thus we may suppose $\gamma = \bar\gamma$, $\delta = \bar\delta$, and so

$$\begin{vmatrix} \rho_{1\xi} & a - \bar a \\ \rho_{2\xi} & c - \bar c \end{vmatrix} = 0, \qquad \begin{vmatrix} \rho_{1\xi} & b - \bar b \\ \rho_{2\xi} & d - \bar d \end{vmatrix} = 0.$$

As $\{\rho_{1\xi}, \rho_{2\xi}\}$ form a regular sequence $a - \bar a = v_1\rho_{1\xi}$, $c - \bar c = v_1\rho_{2\xi}$, $b - \bar b = v_2\rho_{1\xi}$, $d - \bar d = v_2\rho_{2\xi}$ for some $v_1, v_2 \in k$. Subtracting the first column of φ multiplied with v_1 from the second one, we reduce to the case $a = \bar a$, $c = \bar c$. Similarly, subtracting the first column of φ multiplied with v_2 from the third one, we reduce to the case $b = \bar b$, $d = \bar d$, that is, $\varphi = \alpha_\lambda$. $\square$

Theorem 3.8. *The modules of $\mathcal{M}_0 \cup \{R\}$ (see 3.7) induce the bundles of $\mathcal{E}(1,0)$ and the modules of $\mathcal{M}_1, \mathcal{M}_{-1}$ induce the bundles of $\mathcal{E}(1,1), \mathcal{E}(1,-1)$ after some possible shifting.*

Proof. It is enough to see that the graded MCM R-modules of $\mathcal{M}_1$ correspond, after a possible shifting to the bundles of $\mathcal{E}(1,1)$ and conversely. Indeed, then the graded MCM R-modules of $\mathcal{M}_{-1}$, that is, the duals of the graded MCM R-modules of $\mathcal{M}_1$ (see Theorem 3.1(3)) must correspond after a possible shifting to the duals of the bundles of $\mathcal{E}(1,1)$, that is, to the bundles of $\mathcal{E}(1,-1)$. Since $\mathcal{M}_0$ consists of all rank one graded MCM R-modules which are not in $\mathcal{M}_1 \cup \mathcal{M}_{-1}$

we conclude that the modules of $\mathcal{M}_0$ must correspond, after a possible shifting with the bundles of $\mathcal{E}(1,0)$.

By [8, (II, 6.11), (II, 6.15)] any line bundle of degree one has the form $\mathcal{O}_X(P)$ for a point $P \in X$. By [8, (II, 6.18)] (see also the proof of [8, (IV, 1.3)]) the structure sheaf $k(P)$ of the closed sub-scheme $\{P\}$ of X (a skyscraper sitting at P) is given by an exact sequence

$$0 \to \mathcal{O}_X(-P) \to \mathcal{O}_X \to k(P) \to 0.$$

Tensoring with $\mathcal{O}_X(P)$ we obtain a new exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(P) \to k(P) \to 0,$$

since $\mathcal{O}_X(P)$ is locally free of rank one, tensoring by it does not affect $k(P)$.

This new exact sequence is the bottom line of the following commutative diagram with lines and columns exact:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}(-3) \arrow[r] \arrow[d, "f"'] & \mathcal{O}(-3) \oplus \mathcal{O}(-2)^2 \arrow[r] \arrow[d, dashed, "\gamma"] & \mathcal{O}(-2)^2 \arrow[r] \arrow[d, "(\ell_1\,\ell_2)"] & 0 \\
0 \arrow[r] & \mathcal{O} \arrow[r] \arrow[d] & \mathcal{O} \oplus \mathcal{O}(-1) \arrow[r] \arrow[d, dashed] & \mathcal{O}(-1) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{O}_X \arrow[r] \arrow[d] & \mathcal{O}_X(P) \arrow[r] \arrow[d] & k(P) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

where $\mathcal{O} = \mathcal{O}_{\mathbb{P}^2_k}$, the first two lines are canonically split sequences, the first and third columns are parts of the free resolutions of $\mathcal{O}_X$, respectively $k(P)$ over $\mathcal{O}$ (the last one is the Koszul complex and ℓ_1, ℓ_2 are the linear forms defining P, for instance, if $P = (\lambda_1 : \lambda_2 : 1)$ we may take $\ell_1 = Y_1 - \lambda_1 Y_3$, $\ell_2 = Y_2 - \lambda_2 Y_3$) and the second column is constructed canonically. Then γ must be given by a matrix $\begin{pmatrix} f & q_1 & q_2 \\ 0 & \ell_1 & \ell_2 \end{pmatrix}$, where q_i are forms in Y of degree 2.

Let M_P be the graded MCM R-module corresponding to $\mathcal{O}_X(P)$. Tensoring the second column with $\mathcal{O}_X \otimes_{\mathcal{O}}$—see the diagram above—we obtain a corresponding exact sequence

$$R(-3) \oplus R(-2)^2 \xrightarrow{\tau} R \oplus R(-1) \to M_P \to 0,$$

where $\tau = \begin{pmatrix} 0 & q_1 & q_2 \\ 0 & \ell_1 & \ell_2 \end{pmatrix}$. As rank $M_P = 1$ we see that the rows of τ must be linearly dependent and so $\begin{vmatrix} q_1 & q_2 \\ \ell_1 & \ell_2 \end{vmatrix}$ must be a multiple of f with a non-zero constant $u \in k$ by degree reason. Thus, $\begin{vmatrix} u^{-1}q_1 & u^{-1}q_2 \\ \ell_1 & \ell_2 \end{vmatrix} = f$ and so $M_P \cong \operatorname{Coker} \psi_P \in \mathcal{M}_1$, where ψ_P is given in Theorem 3.1. $\square$

Corollary 3.9. *Define $\mathcal{M}_{-1}, \mathcal{M}_1$, and $\mathcal{M}_0$ for $\widehat{R} = k[\![Y_1, Y_2, Y_3]\!]/(Y_1^3 + Y_2^3 + Y_3^3)$ similarly by the corresponding matrix factorizations. Then, every rank one* *maximal Cohen–Macaulay module over $\widehat{R}$ is isomorphic to a module in $\mathcal{M}_{-1} \cup \mathcal{M}_1 \cup \mathcal{M}_0 \cup \{R\}$.*

Proof. The corollary is an immediate consequence of the fact that every reflexive $\widehat{R}$-module is gradable [4, Proposition 5.23], the fact that every rank one maximal Cohen–Macaulay module over $\widehat{R}$ is generated by at most three elements (cf. [6, Corollary 1.3]), and Theorems 3.1 and 3.7. □

4. Cohen–Macaulay modules of higher rank

In this section we use Atiyah's classification to describe the MCM of rank 2 and give an algorithm to compute the matrix factorization of modules of higher rank.

Definition 4.1. For $d = 0, \pm 1, \pm 2, 3$ let $\mathcal{M}(2,d)$ be the set of all isomorphic classes of MCM over R, respectively $\widehat{R}$, corresponding to the vector bundles in $\mathcal{E}(2,d)$.

The idea of the classification is now to describe $\mathcal{M}(2,d)$ using Lemmas 1.3, 1.4, and 1.6.

Theorem 4.2.

(1) *Let M be an indecomposable graded MCM of rank 2 over R, then $M(n) \in \mathcal{M}(2,d)$ for suitable n and d, $-2 \leqslant d \leqslant 3$. Let M be an indecomposable MCM of rank 2 over $\widehat{R}$, then $M \in \mathcal{M}(2,d)$ for a suitable d, $-2 \leqslant d \leqslant 3$.*

(2) *$\mathcal{M}(2,0) = \{(M_2 \otimes L)^{**} \mid L \in \mathcal{M}_0\} \cup \{M_2\}$, where M_2 is given in Theorem* 2.6.

(3) $\mathcal{M}(2,\pm 2) = \left\{(M_2 \otimes L)^{**} \mid L \in \mathcal{M}_{\pm 1}\right\}$.

(4) $$\mathcal{M}(2,3) = \left\{\left(\Omega_R^1(M_2) \otimes L\right)^{**} \mid L \in \mathcal{M}_0\right\} \cup \left\{\Omega_R^1(M_2)\right\} = \left\{\Omega_R^1(L) \mid L \in \mathcal{M}_0\right\} \cup \left\{\Omega_R^1(M_2)\right\}.$$

(5) $$\mathcal{M}(2,\pm 1) = \left\{\left(\Omega_R^1(M_2) \otimes L\right)^{**} \mid L \in \mathcal{M}_{\pm 1}\right\} = \left\{(A \otimes L)^{**} \mid A \in \mathcal{M}(2,3), L \in \mathcal{M}_{\pm 1}\right\}.$$ [4]

Proof. (2) and (3) are immediate consequences of Atiyah's result (Lemma 1.3) and Theorem 3.8. The first equalities of (4) and (5) and the second equality of (5) are consequences of Lemmas 1.6 and 1.4 and the fact that using

[4] The tensor product and the reflexive hull can be computed using SINGULAR (see Section 5) and, therefore, we can obtain all the matrix factorizations of rank 2 MCMs.

Lemma 1.7 $\Omega_R^1(M_2) \in \mathcal{M}(2,3)$. The second equality of (4) follows from the fact that the modules in $\mathcal{M}(2,3)$, except $\Omega_R^1(M_2)$, are generated by three elements (cf. Lemma 5.4). □

Remark 4.3. To give an explicit description of the MCMs of higher rank we again use Atiyah's classification and the fact that we can compute (cf. Section 5)

- $M_r = S^{r-1}(M_2)^{**}$.
- $\Omega_R^1(E) \in \mathcal{M}(r,d')$ for $E \in \mathcal{M}(s,d)$ with $s+r$ generators and a suitable d' with $d+d' \equiv 0(3)$.
- $(E \otimes L)^{**}$ for $L \in \mathcal{M}_1 \cup \mathcal{M}_{-1} \cup \mathcal{M}_0$.

5. Some results obtained by SINGULAR

In this section we want to give the proof for Theorem 3.7(2) and some other useful results we obtained with the help of the computer algebra system SINGULAR [16].

Lemma 5.1. *Let*

$$A = \begin{pmatrix} 0 & y_1 - ay_3 & y_2 - by_3 \\ y_1 + y_3 & -y_2 - by_3 & -zy_3 \\ y_2 & zy_3 & -y_1 + (-a+1)y_3 \end{pmatrix} \quad \text{and}$$

$$B = \begin{pmatrix} 0 & y_1 - cy_3 & y_1 - dy_3 \\ y_1 + y_3 & -y_2 - dy_3 & -xy_3 \\ y_2 & xy_3 & -y_1 + (-c+1)y_3 \end{pmatrix}$$

be two matrices such that $z = b^2/(a+1)$, $x = d^2/(c+1)$, $a^3 + b^3 + 1 = 0$, and $c^3 + d^3 + 1 = 0$. Then A and B are equivalent if and only if $a = c$ and $b = d$.[5]

Proof. We write the conditions $UA = VB$ for suitable invertible matrices U, V: let

$$U = \begin{pmatrix} u_1 & u_2 & u_3 \\ u_4 & u_5 & u_6 \\ u_7 & u_8 & u_9 \end{pmatrix}, \qquad V = \begin{pmatrix} v_1 & v_2 & v_3 \\ v_4 & v_5 & v_6 \\ v_7 & v_8 & v_9 \end{pmatrix},$$

then we obtain the following system of equations:

$$\begin{aligned} &u_2 - v_4 = 0, \qquad && u_8 + v_7 = 0, \qquad && u_6 + v_4 = 0, \\ &u_1 - v_5 = 0, && u_7 + v_8 = 0, && u_5 + v_5 = 0, \end{aligned}$$

[5] Note that A and B define modules of $\mathcal{M}_0$ corresponding to the points $[a:b:1]$, respectively $[c:d:1]$.

$$u_3 - v_6 = 0, \qquad u_9 - v_9 = 0, \qquad u_4 + v_6 = 0,$$
$$u_5 - v_1 = 0, \qquad u_3 - v_7 = 0, \qquad u_9 - v_1 = 0,$$
$$u_4 - v_2 = 0, \qquad u_2 + v_8 = 0, \qquad u_8 + v_2 = 0,$$
$$u_6 + v_3 = 0, \qquad u_1 - v_9 = 0, \qquad u_7 - v_3 = 0,$$
$$u_2 + cv_4 + dv_7 = 0,$$
$$au_1 + bu_2 - wu_3 - cv_5 - dv_8 = 0,$$
$$bu_1 + wu_2 + au_3 - u_4 - cv_6 - dv_9 = 0,$$
$$u_5 - v_1 + dv_4 + xv_7 = 0,$$
$$au_4 + bu_5 - wu_6 + v_2 - dv_3 - xv_8 = 0,$$
$$bu_4 + wu_5 + au_6 - u_6 + v_3 - dv_6 - xv_9 = 0,$$
$$u_8 - xv_4 + cv_7 - v_7 = 0,$$
$$au_7 + bu_8 - wu_9 + xv_5 - cv_8 + v_8 = 0,$$
$$bu_7 + wu_8 + au_9 - u_9 + xv_6 - cv_9 + v_9 = 0,$$
$$\det(U) - 1 = 0, \qquad \det(V) - 1 = 0.$$

It is not difficult to transform this system of equations to the following equivalent system:

$$b = d, \qquad a = c, \qquad c^2 - c + 1 = -wd, \qquad d^2 = w(c+1),$$
$$x = w, \qquad v_9^3 = 1, \qquad v_9 = v_1 = v_5 = u_1 = u_5 = u_9,$$
$$u_2 = u_3 = u_4 = u_6 = u_7 = u_8 = v_2 = v_3 = v_4 = v_6 = v_7 = v_8 = 0,$$

which proves Lemma 5.1.

One can use SINGULAR as follows to perform the transformation of the system above:

```
LIB"matrix.lib";
option(redSB);
ring R=0,(u(1..9),v(1..9),y(1..3),x,w,a,b,c,d),lp;
ideal I=c3+d3+1,
        xd+c2-c+1,
        xc+x-d2,
        a3+b3+1,
        wb+a2-a+1,
        wa+w-b2;
qring Q=std(I);

matrix U[3][3]=u(1..9);
matrix V[3][3]=v(l..9);
matrix A[3][3]=0,         y(1)-a*y(3),  y(2)-b*y(3),
              y(1)+y(3), -y(2)-b*y(3), -w*y(3),
```

```
               y(2),      w*y(3),       -y(1)+(-a+1)*y(3);

matrix B[3][3]=0,         y(1)-c*y(3),  y(2)-d*y(3),
               y(1)+y(3), -y(2)-d*y(3), -x*y(3),
               y(2),      x*y(3),       -y(1)+(-c+1)*y(3);

matrix C=U*A-B*V;

ideal I=flatten(C);
ideal I1=transpose(coeffs(I,y(l)))[2];
ideal I2=transpose(coeffs(I,y(2)))[2];
ideal I3=transpose(coeffs(I,y(3)))[2];
ideal J=I1+I2+I3+ideal(det(U)-1,det(V)-1);
ideal K=std(J);
K;

K[1]=b-d
K[2]=a-c
K[3]=w*d+c^2-c+1
K[4]=w*c+w-d^2
K[5]=x-w
K[6]=v(9)^3-1
K[7]=v(8)
K[8]=v(7)
K[9]=v(6)
K[10]=v(5)-v(9)
K[11]=v(4)
K[12]=v(3)
K[13]=v(2)
K[14]=v(1)-v(9)
K[15]=u(9)-v(9)
K[16]=u(8)
K[17]=u(7)
K[18]=u(6)
K[19]=u(5)-v(9)
K[20]=u(4)
K[21]=u(3)
K[22]=u(2)
K[23]=u(1)-v(9)
```

We see that $b = d$ and $a = c$. □

Lemma 5.2. *Let*

$$A = \begin{pmatrix} 0 & y_1 & y_2 + y_3 \\ y_1 + y_3 & -y_2 + y_3 & y_1 \\ y_2 & -y_1 + y_3 & -y_1 \end{pmatrix}$$

and M the MCM corresponding to A, then $(M \otimes M \otimes M)^* \cong R$.

Let

$$B = \begin{pmatrix} 0 & -y_1 + y_3 & y_2 - by_3 \\ y_1 + y_3 & -y_2 - by_3 & -b^2/2y_3 \\ y_2 & b^2/2y_3 & -y_1 \end{pmatrix}$$

such that $b^3 = -2$ and N the MCM corresponding to B, then N is self-dual.

Proof. First of all we give procedures to compute the reflexive hull, the tensor product in the category of Cohen–Macaulay modules, the module M_2 and to check the isomorphy of two MCMs, which are generated by three or six elements.

```
LIB"matrix.lib";
proc reflexivHull(matrix M)
{
  module N=mres(transpose(M),3)[3];
  N=prune(transpose(N));
  return (matrix(N));
}

proc tensorCM(matrix Phi, matrix Psi)
{
  int s=nrows(Phi);
  int q=nrows(Psi);
  matrix A=tensor(unitmat(s),Psi);
  matrix B=tensor(Phi,unitmat(q));
  matrix R=concat(A,B);
  return(reflexivHull(R));
}

proc M2(ideal I)
{
  matrix A=syz(transpose(mres(I,3)[3]));
  return (transpose (A));
}

proc isIsoCM(matrix A,matrix B)
{
  def R=basering;
  int n=nrows(A);
  int m=n*n;
  number p;

  if (deg(minpoly)!=-1){p=minpoly;}
  execute("ring S=("+charstr(R)+"),("+varstr(R)+",
                               u(1.."+string(m)+"),
                               v(1.."+string(m)+")),dp;");
```

```
  number p=imap(R,p);
  if(deg(p)!=-l){minpoly=p;}

  matrix U[n][n]=u(l..m);
  matrix V[n][n]=v(1..m);
  matrix A=imap(R,A);
  matrix B=imap(R,B);
  matrix C=U*A-B*V;
  module se=syz(ideal(det(A),det(B)));
  ideal I=flatten(C);
  int j;
  ideal J=det(U)-se[1][1],det(V)+se[1][2];
  for(j=1;j<=size(I);j++)
  {
    J=J+transpose(coef(I[j],var(1)*var(2)*var(3)))[2];
  }
  int d=deg(std(J)[1]);
  setring R;
  if (d==0){return (0);}
  return(1);
}

ring R=(0,b),(y(1..3)),(c,dp);
minpoly=b3+2;
qring S=std(y(1)^3+y(2)^3+y(3)^3);

matrix A[3][3]=0,          y(1),          y(2)+y(3),
               y(1)+y(3), -y(2)+y(3),   y(1),
               y(2),      -y(1)+y(3),   -y(1);

matrix B[3][3]=0,          y(1)-y(3),    y(2)-b*y(3),
               y(1)+y(3), -b*y(3)-y(2), -(b*b/2)*y(3),
               y(2),      (b*b/2)*y(3), -y(1);

tensorCM(A,tensorCM(A,A));
_[1,1]=0
```

This proves that $(M \otimes M \otimes M)^* \cong R$.

```
tensorCM(B,B);
_[1,1]=0
```

We obtain that $(N \otimes N)^* \cong R$. $\square$

Lemma 5.3. *Let M and N be defined as in Lemma* 5.2. *Then the following hold*:

(1) $(\Omega_R^2(\mathfrak{m}) \otimes N)^{**}$ *is not isomorphic to* $\Omega_R^2(\mathfrak{m})$.

(2) $\Omega_R^1(\mathfrak{m}) \cong (\Omega_R^1(M) \otimes M^*)^{**}$.[6]
(3) $(\Omega_R^1(\mathfrak{m}) \otimes N)^{**} \cong \Omega_R^1(\mathfrak{m})$.
(4) $0 \to R \to \Omega_R^1(\mathfrak{m}) \to \langle y_1^2, y_2, y_3 \rangle \to 0$ *is a Bourbaki exact sequence for* $\Omega_R^1(\mathfrak{m})$.
(5) $\Omega_R^2(\mathfrak{m})$ *is isomorphic to the reflexive hull of the differential module of* R.

Proof.

```
ideal I=maxideal(1);
matrix C=M2(I);
print(C);

0,       y(2),    y(1),     y(3)^2,
y(2),    0,       y(3),     -y(1)^2,
y(1),    -y(3),   0,        y(2)^2,
y(3)^2, y(1)^2, -y(2)^2, 0
```

This is the matrix corresponding to $M_2 = \Omega_R^2(\mathfrak{m})$.

```
matrix C1=tensorCM(C,B);
```

This is the matrix corresponding to $(M_2 \otimes N)^{**}$.

```
nrows(Cl);
6
nrows (C);
4
```

The module $(\Omega_R^2(\mathfrak{m}) \otimes N)^{**}$ is generated by six elements. The module $\Omega_R^2(\mathfrak{m})$ is generated by four elements. They cannot be isomorphic. This proves (1).

```
matrix D=transpose(syz(C));
```

D is the matrix corresponding to $\Omega_R^1(\mathfrak{m})$.

```
matrix E=tensorCM(D,A);
```

E is the matrix corresponding to $(\Omega_R^1(\mathfrak{m}) \otimes M)^{**}$.

```
matrix F=syz(A);
```

F is the matrix corresponding to $\Omega_R^1(M)$.

```
isIsoCM(F,E);
1
```

This proves (2).

```
matrix E1=tensorCM(B,E);
```

E1 is the matrix corresponding to $(\Omega_R^1(\mathfrak{m}) \otimes M \otimes N)^{**}$.

[6] M is as rank 1 module indecomposable and, therefore, $\Omega_R^1(M)$ and $(\Omega_R^1(M) \otimes M^*)^{**}$ are indecomposable, too. This is another proof for the fact that $\Omega_R^1(\mathfrak{m})$ and $\Omega_R^2(\mathfrak{m})$ are indecomposable.

```
isIsoCM(E1,E);
1
```

This proves (3).

```
I=y(1)^2,y(2),y(3);
matrix D1=M2(I);
E1=tensorCM(D1,A);
```

This is the matrix corresponding to the module defined by the Bourbaki sequence of (4), tensorised by M.

```
isIsoCM(E1,E);
1
```

This proves (4).

```
ring R0=0,(y(1..3)),(c,dp);
qring q=std(y(1)^3+y(2)^3+y(3)^3);
ideal I=jacob(y(1)^3+y(2)^3+y(3)^3);

matrix E=reflexivHull(transpose(matrix(I)));
```

The matrix corresponding to the reflexive hull of the differential module of R.

```
 print(E);

0,        y(3),     y(2),     y(1)^2,
y(3),     0,        -y(1),    y(2)^2,
y(2),     y(1),     0,        -y(3)^2,
y(1)^2, -y(2)^2, y(3)^2, 0

I=maxideal(1);
matrix C=F2(I);
matrix A=imap(R,A);

matrix E1=tensorCM(E,A);
matrix C1=tensorCM(C,A);
```

We tensorise both modules by M (corresponding to the matrix A) to obtain modules generated by six elements. They are easier to compare.

```
isIsoCM(E1,C1);
1
```

This proves (5). □

Lemma 5.4.

(1) *The MCM corresponding to* $\mathcal{M}(2,0)$ *except* M_2 *are generated by six elements.*

(2) *The MCM corresponding to $\mathcal{M}(2,2), \mathcal{M}(2,-2)$ are generated by four elements.*

(3) *The MCM corresponding to $\mathcal{M}(2,3)$ except $\Omega^1_R(M_2)$ are generated by three elements.*

(4) *The MCM corresponding to $\mathcal{M}(2,-1), \mathcal{M}(2,1)$ are generated by five elements.*

Proof.

```
ring R1=(0,a),(y(1..3),b),(c,1p);
ideal I=y(1)^3+y(2)^3+y(3)^3,
        a3+b3+1;

qring S1=std(I);

matrix A[2][2]=y(3)-a*y(1),    y(2)^2+b*y(2)*y(1)+b^2*y(1)^2,
               -(y(2)-b*y(1)), y(3)^2+a*y(1)*y(3)+a^2*y(1)^2;
matrix A1[2][2]=y(1)+y(3), y(2)^2,
                -y(2),     y(1)^2-y(1)*y(3)+y(3)^2;

matrix C=imap(S,C);
```

The matrix corresponding to M_2.

```
matrix D=imap(S,D);
```

The matrix corresponding to $\Omega^1_R(M_2)$.

```
nrows(tensorCM(C,A));
4
nrows(tensorCM(transpose(A),C);
4
```

This proves (2).

```
nrows(tensorCM(D,A));
5
nrows(tensorCM(transpose(A),D));
5
```

This proves (4).

```
matrix D1=tensorCM(D,transpose(A1));
print(D1);

y(1)-y(3), 0,            y(2),      -y(3), y(2)^2,
0,         y(1)-2*y(3), 0,          -y(2), -3*y(3)^2,
0,         -y(2),        y(1)+y(3), 0,     -y(2)*y(3),
y(3),      0,            y(2),      y(1),  0,
-y(2), 3*y(3),  0, y(2),  y(1)^2+2*y(1)*y(3)+4*y(3)^2
```

This is a special element in $\mathcal{M}(2,1)$. Now we use the fact that $\mathcal{M}(2,\pm 1) = \{(A \otimes L)^{**} \mid A \in \mathcal{M}(2,3), L \in \mathcal{M}_{\pm 1}\}$.

```
nrows (tensorCM(D1,A));
3
```

This proves (3).

```
ring R2=(0,a),(y(1..3),e,b),1p;
ideal I=y(1)^3+y(2)^3+y(3)^3,
        a3+b3+1,
        e*b+a2-a+1,
        e*a+e-b2;

qring S2=std(I);

matrix B[3][3]=0,          y(3)-a*y(1),  y(2)-b*y(1),
               y(3)+y(1), -b*y(1)-y(2), -e*y(1),
               y(2),       e*y(1),       (-a+1)*y(1)-y(3);

matrix C=imap(S,C);
nrows (tensorCM(C,B));
6
```

This proves (1). □

Finally, we give the matrix factorizations for M_3 and M_4. Here we use the following description of the symmetric algebra: Let $A = (a_{ij})$ be the $m \times n$ presentation matrix of the R-module M and

$$\begin{pmatrix} F_1(z) \\ \vdots \\ F_m(z) \end{pmatrix} = A \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix};$$

let $S := R[z_1, \ldots, z_n]/(F_1, \ldots, F_m)$, then S is the symmetric algebra of M and $S_n = \{H \in S \mid H \text{ homogeneous in } z, \deg_z H = n\}$ is the nth symmetric power of M. The corresponding reflexive module is $S^n(M) = S_n^{**}$. We use the following procedure:

```
proc sym(matrix M,int n)
{
  def R=basering;
  int m=ncols(M);
  string s=string(m);
  number p;
  int j;

  if(deg(minpoly)!=-1){p=minpoly;}

  execute("ring S=
```

```
  ("+charstr(R)+"),("+varstr(R)+",z(1.."+s+")),dp;");
  number p=imap(R,p);
  if(deg(p)!=-1){minpoly=p;}
  matrix M=imap(R,M);
  matrix N[m][1]=z(1..m);
  ideal K=z(1..m);
  ideal I=flatten(M*N)*(K^(n-1));
  K=K^n;
  poly f=z(1);
  for(j=2;j<=m;j++)
  {
     f=f*z(j);
  }
  matrix T=coeffs(I,K,f);
  setring R;
  matrix T=imap(S,T);
  return(reflexivHull(T));
}


ring R=0,(y(1..3)),dp;
qring Q=std(y(1)^3+y(2)^3+y(3)^3);
ideal I=maxideal(1);
matrix C=M2(I);
print(sym(C,2));

0,                  y(1),   0,   -y(2),   0,   -y(3), 0,
0,                 -y(3),   y(1), 0,      0,    0,   -y(2),
0,                  0,      0,    y(3),  -y(2), 0,    y(1),
2*y(3)^2,           0,      y(2), 0,      y(1), 0,    0,
2*y(2)^2,           0,     -y(3), 0,      0,   -y(1), 0,
2*y(1)^2,           0,      0,    0,     -y(3), y(2), 0,
2*y(1)*y(2)*y(3), y(2)^2, 0,    y(1)^2, 0,    0,   -y(3)^2

print(sym(C,3));

0,       y(1), -y(2), 0,   0,    0,    0,    0, -y(3),  0,
0,        0,    0,    0, -y(2),  0,   y(1),  0,    0,-2*y(3),
0,        0,    0,    0,   0,    0,  -y(3),2*y(1),y(2), 0,
0,        0,    0,    0,-y(3),-2*y(2), 0,    0,   y(1), 0,
y(3)^2,   0,    0,    0,   0,   y(1),  0, -y(2),   0,   0,
y(2)*y(3),-y(3),0, -y(2),-y(1),  0,    0,    0,    0,   0,
y(1)*y(3),0,  y(3), y(1),  0,    0,  -y(2),  0,    0,   0,
y(2)^2,   0,    0,    0,   0,    0,    0,   y(3),  0, y(1),
y(1)^2,   0,    0,    0,   0,  -y(3),  0,    0,    0,-y(2),
0,   y(2)^2,y(1)^2,-y(3)^2,0,y(1)*y(3),0,y(2)*y(3),0,   0
```

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