Math 535 (Spring 2020)

Homework 2

due February 26

1. On a 2-dimensional k-vector space E consider the bilinear forms b_i ($i = 1, \dots, 4$) with associated Gram matrices

$$M_1 = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}, M_2 = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}, M_3 = \begin{pmatrix} 4 & 2 \\ 2 & 3 \end{pmatrix}, M_4 = \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}$$

   Decide which of the forms b_i are isometric[1] when

   i) $k = \mathbb{Q}$;

   ii) $k = \mathbb{R}$;

   iii) $k = \mathbb{C}$.

2. Let k be a field with $\mathrm{char}(k) \neq 2$, and E a four-dimensional k-vector space. Assume that E comes equipped with a non-degenerate symmetric bilinear form $(-,-)$ such that there exist vectors $v_1, v_2, w \in E$ with

   - $(w, w) = 0$, $w \perp v_i$ for $i = 1, 2$;
   - $(v_1, v_1) = 12$, $(v_2, v_2) = -2$, $(v_1, v_2) = 1$.

   What is the Witt normal form of E?

3. Let $E = \mathbb{R}^3$ with the standard inner product. Find an orthonormal basis of eigenvectors for the matrix

$$M = \begin{pmatrix} 3 & -2 & 4 \\ -2 & 6 & 2 \\ 4 & 2 & 3 \end{pmatrix}.$$

   Find a matrix B such that $B^{-1} = B^t$ and $B^{-1}MB$ is diagonal.

[1] Recall, $M' \sim M$ if $M' = B^t M B$ for some B

4. For a matrix A, define the exponential

$$e^A := I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \dots.$$

Compute e^A for

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

5. Let V be a finite dimensional $\mathbb{C}$-vector space endowed with a Hermitian form h. Let $W = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}(V)$, i.e. V viewed as an $\mathbb{R}$ vector space. On W define the $\mathbb{R}$-bilinear forms

$$\begin{aligned} g(x,y) &= \mathrm{Re}(h(x,y)) \\ \omega(x,y) &= \mathrm{Im}(h(x,y)) \end{aligned}$$

Prove that g is symmetric bilinear, while ω is alternating. Show that given g [2], you can recover h and ω. [Similarly, ω recovers h and g]

6. With notations as in the previous exercise. Assume that the Hermitian form h has Gram matrix

$$\begin{pmatrix} 1 & 2+3i \\ 2-3i & 4 \end{pmatrix}$$

What are the Gram matrices for g and ω respectively?

7. Let V be a finite dimensional $\mathbb{C}$-vector space with Hermitian inner product and let A, B be commuting self-adjoint operators on V. Prove that A and B have a common orthonormal eigenbasis.

8. Let $A = \begin{pmatrix} 0 & -2 \\ 1 & 0 \end{pmatrix}$. Find the polar decomposition of A:

$$A = UP$$

with U a unitary matrix and P positive definite (see Lang XV, Thm. 6.9).

[2] and secretly the identification between V and W