THE LLV DECOMPOSITION OF HYPER-KÄHLER COHOMOLOGY

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ABSTRACT. Looijenga–Lunts and Verbitsky showed that the cohomology of a compact hyper-Kähler manifold \( X \) admits a natural action by the Lie algebra \( \mathfrak{so}(4, b_2(X) - 2) \), generalizing the Hard Lefschetz decomposition for compact Kähler manifolds. In this paper, we determine the Looijenga–Lunts–Verbitsky (LLV) decomposition for all known examples of compact hyper-Kähler manifolds. As an application, we compute the Hodge numbers of the exceptional OG10 example starting only from the knowledge of the Euler number \( e(X) \), and the vanishing of the odd cohomology of \( X \). In a different direction, we establish the so-called Nagai’s conjecture for all known examples of hyper-Kähler manifolds. More importantly, we prove that, in general, Nagai’s conjecture is equivalent to a representation theoretic condition on the LLV decomposition of the cohomology \( H^*(X) \). We then notice that all known examples of hyper-Kähler manifolds satisfy a stronger, more natural condition on the LLV decomposition of \( H^*(X) \): the Verbitsky component is the dominant representation in the LLV decomposition of \( H^*(X) \).

1. Introduction

The compact hyper-Kähler manifolds are one of the most interesting building blocks in algebraic and complex geometry, as they are the most likely case to admit a good general classification. Indeed, they are \( K \)-trivial varieties, and among the three possible irreducible pieces for \( K \)-trivial varieties, they occupy the middle ground between complex tori (trivial classification) and Calabi-Yau manifolds (already too varied in dimension 3). Unfortunately, all that is known so far is a small list of examples of hyper-Kähler manifolds: two infinite series, \( K3^{[n]} \) and \( \text{Kum} \), in dimension \( 2n \), due to Beauville [Bea83], and two exotic examples, OG10 and OG6 in dimension 10 and 6 respectively, due to O’Grady [O’G99, O’G03]. Not only it is not known if this list is essentially complete, but even the finiteness of the deformation types in any dimension \( 2n(> 2) \) is a wide open question.

Verbitsky’s Global Torelli Theorem [Ver13] says that a hyper-Kähler manifold \( X \) is determined up to birational equivalence by the Hodge structure on the second cohomology \( H^2(X) \). While this is similar to saying that a complex torus \( A \) is determined by \( H^1(A) \), in contrast to the case of tori, the reconstruction of \( X \) from its second cohomology \( H^2(X) \) is very mysterious. In this paper, we focus on reconstructing the entire cohomology \( H^*(X) \) from the second cohomology \( H^2(X) \) (at least in the known examples mentioned above). Our starting point is the work of Verbitsky [Ver90, Ver95, Ver96] and Looijenga–Lunts [LL97] who have noted that, for hyper-Kähler manifolds, \( H^*(X) \) admits a natural representation by the Lie algebra \( \mathfrak{g} = \mathfrak{so}(4, b_2(X) - 2) \), generalizing the usual \( \mathfrak{sl}(2) \) representation that occurs in the Hard Lefschetz Theorem. We call this Lie algebra \( \mathfrak{g} \) the Looijenga–Lunts–Verbitsky (LLV) algebra of \( X \). The LLV algebra \( \mathfrak{g} \) is determined by the second cohomology. Namely, \( \mathfrak{g} \) is the special orthogonal algebra associated to the quadratic space \( V := (H^2(X, \mathbb{R}), q_X) \oplus U \), where \( q_X \) is the Beauville–Bogomolov–Fujiki quadratic form on \( H^2(X) \), and \( U \) is the standard hyperbolic plane (\( V \) is the Mukai completion of \( H^2(X) \)). Importantly, note that \( \mathfrak{g} \) is a semi-simple Lie algebra defined over \( \mathbb{Q} \). By construction, the resulting decomposition, referred throughout as the LLV decomposition, of \( H^*(X) \) into irreducible \( \mathfrak{g} \) representations is a diffeomorphism invariant of \( X \). Furthermore, all natural decompositions of the cohomology \( H^*(X) \) factor through the LLV decomposition. Here the examples of such decompositions are the Hodge decomposition once a complex structure is fixed, the usual \( \mathfrak{sl}(2) \)-Lefschetz decomposition once a Kähler form is fixed, and Verbitsky’s \( \mathfrak{so}(4,1) \)-decomposition once a hyper-Kähler metric is fixed.

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The existence of the LLV decomposition has strong consequences on the cohomology of hyper-Kähler manifolds. For instance, Verbitsky has described explicitly the subalgebra of $H^*(X)$ generated by $H^2(X)$, and for many questions this knowledge suffices. From our perspective, we state Verbitsky’s result as saying that for a $2n$-dimensional hyper-Kähler manifold $X$, the irreducible $g$-representation generated by $H^2(X)$ in $H^*(X)$ is $V_n = V_{n, \bar{w}_1} \subset \text{Sym}^n V$ (where $\bar{w}_1$ is the fundamental weight for the standard representation $V = V_{\bar{w}_1}$ of $g \cong \mathfrak{so}(4, b_2 - 2)$)\footnote{We will use the notation $V_\mu$ for the irreducible $g$-module with highest weight $\mu = (\mu_1, \ldots, \mu_r + 1)$. Occasionally, for reader’s convenience, we will write our results in terms of the fundamental weights $\bar{w}_i$. We refer the reader to the Appendix (esp. (A.1) and (A.5)) for the precise translation formula between the notations $\bar{w}_i$ and $(\mu_1, \ldots, \mu_r + 1)$.}. We call it the Verbitsky component. The Verbitsky component $V_{(n)}$ is always present (with multiplicity 1) in the LLV decomposition of $H^*(X)$. The remaining question is what other representations occur in the LLV decomposition of a hyper-Kähler manifold $X$, and what restrictions do they satisfy. While some general results are established, our paper is primarily concerned with the study of the known cases of hyper-Kähler manifolds $X$, by which we mean $X$ is of $K3^{[n]}$, $\text{Kum}_n$, $\text{OG}_6$, or $\text{OG}_10$ type.

The Betti and Hodge numbers of all the known cases of hyper-Kähler manifolds were previously worked out by other authors. Specifically, Göttsche and Soergel [GS90, GS93] have studied the Hodge structure $H^*(X)$ for the two infinite series $K3^{[n]}$ and $\text{Kum}_n$. More recently, the two exceptional O’Grady cases were settled by Mongardi–Rapagnetta–Saccà [MRS18] for $\text{OG}_6$ type, and by de Cataldo–Rapagnetta–Saccà [dCRS19] for $\text{OG}_10$ type. While these previous results are closely related to the LLV decomposition in these known cases, surprisingly the question of actually describing the LLV decomposition does not seem to have been addressed previously. Our first result does exactly this.

**Theorem 1.1.** The LLV decomposition of the known classes of hyper-Kähler manifolds is as follows:

(i) The generating series of the formal characters of the $\mathfrak{so}(4, 21)$-modules $H^*(K3^{[n]})$ is

$$1 + \left( \sum_{i=0}^{11} (x_i + x_i^{-1}) \right) q + \sum_{n=2}^{11} \text{ch}(H^*(K3^{[n]})) q^n = \prod_{i=0}^{11} \prod_{j=0}^{1} \frac{1}{(1 - x_i q^n)(1 - x_i^{-1} q^n)}.$$\hspace{1cm} (1.2)

(The identity should be understood in the formal power series ring $A[[q]]$, where $A = \mathbb{Z}[x_0, \ldots, x_{11}, (x_0 \cdots x_{11})^{\frac{1}{2}}]^{\mathfrak{w}}$ is the complex representation ring of $\mathfrak{so}(4, 21)$, and $\mathfrak{w}$ indicates the Weyl group of $B_{12}$.)

(ii) Define the formal power series

$$B(q) = \prod_{m=1}^{\infty} \left[ \prod_{j=0}^{3} \prod_{i=0}^{1} \frac{1}{(1 - x_i q^m)(1 - x_i^{-1} q^m)} \prod j (1 + x_0^{b_1} x_1^{j_1} x_2^{j_2} x_3^{j_3} q^m) \right],$$

with $j = (j_0, \cdots, j_3) \in \{-\frac{1}{2}, \frac{1}{2}\}^4$ and $j_0 + \cdots + j_3 \in 2\mathbb{Z}$. Let $b_1$ be the degree 1 coefficient of $B(q) = 1 + b_1 q + b_2 q^2 + \cdots$, and $J_4(d) = d^4 \prod_{p}(1 - \frac{1}{p^d})$ be the fourth Jordan totient function. With these notations, the generating series of the formal characters of the $\mathfrak{so}(4, 5)$-modules $H^*(\text{Kum}_n)$ is

$$1 + \left( \sum_{i=0}^{3} (x_i + x_i^{-1}) + 16 \right) q + \sum_{n=2}^{\infty} \text{ch}(H^*(\text{Kum}_n)) q^n = \sum_{d=1}^{\infty} J_4(d) \frac{B(q^d) - 1}{b_1 \cdot q}.$$\hspace{1cm} (1.3)

(Again, the identity holds in $A[[q]]$ where $A = \mathbb{Z}[x_0, \ldots, x_3, (x_0 \cdots x_3)^{\frac{1}{2}}]^{\mathfrak{w}}$ is the complex representation ring of $\mathfrak{so}(4, 5)$.)

(iii) As a $\mathfrak{so}(4, 6)$-module,

$$H^*(\text{OG6}) = V_{3, \bar{w}_3} \oplus V_{\bar{w}_1, \bar{w}_3} \oplus V_{\bar{w}_3}^{\oplus 135} \oplus \mathbb{R}^{\oplus 240},$$\hspace{1cm} (1.4)

where $V$ is the standard representation, and $\mathbb{R}^{\oplus 240}$ stands for 240 copies of the trivial representation.

(iv) As a $\mathfrak{so}(4, 22)$-module,

$$H^*(\text{OG10}) = V_{5, \bar{w}_5} \oplus V_{2, \bar{w}_2}.$$\hspace{1cm} (1.5)
Remark 1.6. A more compact version of (1.2) is

$$\sum_{n=0}^{\infty} \text{ch}(H^*(\text{K3}[n]))q^n = \prod_{m=1}^{11} \prod_{i=0}^{11} \frac{1}{(1-x_iq^m)(1-x_i^{-1}q^m)},$$

by noting that formally

$$\text{ch}(H^*(\text{K3}[0])) = 1, \quad \text{ch}(H^*(\text{K3}[1])) = \sum_{i=0}^{11} (x_i + x_i^{-1}).$$

The reason for using (1.2) above is that $H^*(\text{K3}[1])$ does not have a structure of $\mathfrak{so}(4,21)$-module. Similar discussion applies also to the second identity (1.3) for $\text{Kum}_n$ hyper-Kähler manifolds.

Remark 1.8. As is often the case for infinite families, it is more convenient to work with the generating series (1.2) and (1.3) to encode the LLV module structure of the cohomology of $\text{K3}[n]$ and $\text{Kum}_n$ types. However, one can also determine their explicit LLV decompositions. We refer to Corollaries 3.2 and 3.6 for these explicit descriptions when $\dim X \leq 10$. Furthermore, the two generating series can be easily specialized to the generating series for the Hodge-Deligne polynomials, Poincaré polynomials, signatures of the middle cohomology, or the Euler numbers. We recover this way some well known formulas of Göttscbe [G90, G94] (see Corollary 3.8 and 3.9 for the $\text{K3}[n]$ and $\text{Kum}_n$ case respectively).

The original motivation for our paper was the seemingly unrelated study of degenerations of hyper-Kähler manifolds and specifically the so-called Nagai conjecture [Nag08]. Let $X/\Delta$ be a one-parameter degeneration of hyper-Kähler manifold. Similar to the K3 case, it is natural to define the Type of the degeneration to be I, II, or III, in accordance to the index of nilpotence $\nu_2$ of the log monodromy operator $N_2 = \log(T_2)_u$ on $H^2(X)$. However, in contrast to the case of K3 surfaces, the hyper-Kähler manifolds $X$ of dimension $2n > 2$ have interesting higher cohomology, and thus, it is natural to investigate its behavior under degenerations. For instance, as a consequence of the fact that the Verbitsky component $V(n) \subset H^*(X)$ controls the holomorphic part of the cohomology, one sees (e.g. [KLSV17, §6.2]) that Type III degenerations of hyper-Kähler manifolds (defined in terms of $H^2$) are equivalent to maximal unipotent monodromy (MUM) degenerations (defined in terms of the middle cohomology $H^{2n}$). More generally, it is natural to expect that the index of nilpotency $\nu_{2k}$ of log monodromy $N_{2k}$ on $H^{2k}(X_i)$ satisfies

$$\nu_{2k} = k \cdot \nu_2 \quad \text{for} \quad k = 1, \ldots, n. \quad (1.9)$$

We refer to (1.9) as Nagai’s conjecture, as Nagai [Nag08] was the first to investigate this type of question, and he established (1.9) for the $\text{K3}[n]$ type and partially for $\text{Kum}_n$ type. Nagai’s conjecture was also verified for all projective degenerations of hyper-Kähler manifolds for Type I and III in [KLSV17], leaving only the Type II case open. Here we establish Nagai’s conjecture in full, including non-projective degenerations, for all known examples of hyper-Kähler manifolds.

Theorem 1.10. Let $X/\Delta$ be a flat proper family, such that the general fiber $X_i$ is a hyper-Kähler manifold of $\text{K3}[n]$, $\text{Kum}_n$, $\text{OG6}$, or $\text{OG10}$ type. Then Nagai’s conjecture (1.9) holds (with $\dim X_i = 2n$).

The approach in [KLSV17] to Nagai’s conjecture is based on studying Kulikov type normalizations for projective degenerations $X/\Delta$. For instance, using some deep results from the minimal model program and specific results on hyper-Kähler geometry, it is shown in loc. cit. that a family $X/\Delta$ with finite monodromy on $H^2(X_i)$ (equivalently $\nu_2 = 0$) can be modified (after base change and birational modifications) to a smooth family $X'/\Delta$. After further arguments, one concludes that $X'/\Delta$ have finite monodromy on all cohomology groups, establishing (1.9) under the Type I assumption. The approach here is essentially orthogonal, focusing exclusively on the cohomological behavior (in particular, our results will say little about the geometric shape of the degeneration). We note that Soldatenkov’s [Sol18] study of odd cohomology under MUM degenerations is similar to ours here.

To start, we consider the interplay between the LLV decomposition of the cohomology $H^*(X)$ and the period map. First, it is well known how the $H^2(X)$ determines the Hodge structure on $H^*(X)$ by means of LLV $g$-representation (see Section 2 below; cf. also [Sol19]). We then lift this to the level of period maps and period domains. Specifically, we prove that for families of hyper-Kähler manifolds, the infinitesimal period map on the second cohomology $H^2(X)$ determines the infinitesimal period map on the entire cohomology...
$H^*(X)$ (Theorem 4.1). It then follows that the log monodromy $N_k$ on the $k^{th}$ cohomology is determined by $N_2$ via the LLV representation, a result previously noticed by Soldatenkov [Sol18, Prop. 3.4] (by a different method). In conclusion, Nagai’s conjecture reduces to a representation theoretic question. Namely, the given data is the nilpotency index of $N_2 \in \mathfrak{g}$ acting on the standard $\mathfrak{g}$-module $\mathcal{V} = H^2(X)$. We are interested in the nilpotency index of $N_{2k} = \rho_k(N_2)$ where $\rho_{2k} : \mathfrak{g} \to \text{End}(H^{2k}(X))$ is the degree $2k$ restriction of the LLV representation. Using a representation theoretic computation we conclude that Nagai’s condition (1.9) is equivalent to the following condition on the dominant $\mu$-weights occurring in the LLV decomposition.

**Proposition 1.11.** Let $X$ be a hyper-Kähler manifold of dimension $2n$ with $b_2(X) \geq 4$. Let $\mathfrak{g} = \mathfrak{so}(4, b_2 - 2)$ be the associated LLV algebra, and

$$H^*_{\text{even}}(X) \cong \bigoplus_{\mu \in S} V^\mu \oplus m\rho_r,$$  

(1.12)

be the decomposition of the cohomology of $X$ into irreducible $\mathfrak{g}$-representations (where $\mu = (\mu_0, \cdots, \mu_r)$ indicates a dominant integral weight of $\mathfrak{g}$, $r = \lfloor b_2(X)/2 \rfloor$, and $V_\mu$ the irreducible $\mathfrak{g}$-module of highest weight $\mu$). Then Nagai’s condition (1.9) holds if and only if every highest weight $\mu \in S$ in (1.12) satisfies

$$\mu_0 + \mu_1 + |\mu_2| \leq n.$$  

(1.13)

Using this criterion and our computation of the LLV decomposition in known cases (Theorem 1.1), we conclude that Nagai’s conjecture’s holds in all known examples of hyper-Kähler manifolds (Theorem 1.10). While we are able to check (1.13) holds under some general (strong) assumptions (e.g. $\dim(X) \leq 8$), we are not able to able to verify (1.13) in general. In fact, we expect (1.13) to be a new condition on the cohomology of hyper-Kähler manifolds (it holds for the known cases, but we believe it to be an open question in general). A more natural condition seems to us to be the stronger condition (1.15) below, which says that the Verbitsky’s component $V_{(n)}$ is the dominant component of the LLV representation of $H^*(X)$ (see Remark 6.3 for the precise meaning of this statement). With not much more effort, we see that all known examples of hyper-Kähler manifolds satisfy this stronger condition. We conclude with a stronger, and perhaps more natural version of Theorem 1.10.

**Theorem 1.14.** Let $X$ be a $2n$-dimensional hyper-Kähler manifold of $K3^{[n]}$, $\text{Kum}_n$, $\text{OG6}$, or $\text{OG10}$ type. Then any dominant integral weight $\mu$ occurring in the LLV decomposition of $H^*(X)$ satisfies

$$\mu_0 + \mu_1 + \cdots + |\mu_r| \leq n.$$  

(1.15)

It is tempting to speculate that (1.15) holds in general, giving a new restriction on the cohomology of hyper-Kähler manifolds. This restriction is strong enough to imply the vanishing of odd cohomology as soon as $\lfloor b_2(X)/2 \rfloor > 2 \dim X - 1$ (Remark 3.30). Thus, a posteriori, we see that Theorem 1.14 implies the vanishing of odd cohomology for $\text{OG10}$, and then recovers the Hodge diamond for $\text{OG10}$ from little geometric data (see Theorem 3.26).

**Structure of the paper.** We start in Section 2 with a discussion of the LLV algebra $\mathfrak{g}$ and its action on the cohomology $H^*(X)$. While this is mostly standard material, we make some small but important observations. For instance, we note the LLV $\mathfrak{g}$ is defined over $\mathbb{Q}$ and describe its $\mathbb{Q}$-algebra structure. This allows us to relate the LLV algebra $\mathfrak{g}$ to the Mumford–Tate (MT) algebra $\mathfrak{m}$. Note that $\mathfrak{g}$ is a diffeomorphism invariant, while on the other hand $\mathfrak{m}$ varies in moduli (depends on the Hodge structure). For hyper-Kähler manifolds, one can enlarge the Lie algebra $\mathfrak{m}$ to its Mukai completion $\mathfrak{m}$. We note that $\mathfrak{m} \subset \mathfrak{g}$, and thus it acts on the cohomology of $X$. Typically, by construction, one understands the Hodge structure on $H^*(X)$, or equivalently the decomposition of $H^*(X)$ with respect to $\mathfrak{m}$. One of our main tools for the proof of Theorem 1.1 (in Section 3) is to use representation theory to lift this $\mathfrak{m}$-representation to a $\mathfrak{g}$-representation.

For concreteness, let us briefly discuss this procedure for a $K3^{[n]}$ type hyper-Kähler manifold $X$. By definition, we can specialize $X$ to the one isomorphic to $S^{[n]}$ for a K3 surface $S$. Then the formula of Göttscbe–Sörgel expresses the cohomology of $H^*(X)$ in terms of the cohomology of $H^*(S)$ as Hodge structures. From our perspective, these results express the cohomology of $H^*(X)$ as a representation of the Mumford–Tate algebra $\mathfrak{m}$. Moreover, taking $S$ as a very general non-projective K3 surface (which is allowed by $[d\text{CM}]$), we can further assume $\mathfrak{m} \cong \mathfrak{so}(3,19)$. Then, one can can easily construct the Mukai completion, which means that, using the natural degree grading on cohomology, one can lift the $\mathfrak{so}(3,19)$-module structure of $H^*(X)$ to a $\mathfrak{so}(4,20)$-module structure. However, we are still not done, as this Lie algebra is still slightly
smaller than the LLV algebra \( so(4, 21) \) for \( K3^{[n]} \). We now use a representation theory fact on restriction representations. Namely, the Lie algebras \( so(4, 20) \) and \( so(4, 21) \) are of type \( D_{12} \) and \( B_{12} \), in particular are of the same rank. It follows that the restriction representation functor \( \text{Rep}(B_{12}) \to \text{Rep}(D_{12}) \) is injective on the level of objects. Thus, there is a unique lift of the \( so(4, 20) \)-module structure on \( H^*(X) \) to a \( so(4, 21) \)-module structure. In other words, we have lifted the Göttsche–Soergel presentation of \( H^*(X) \) as a Hodge structure to the LLV decomposition of \( H^*(X) \).

The Kas case is similar but more complicated as it contains nonvanishing odd cohomology and many trivial representations. For example, to get a flavor of this phenomenon of an excessive amount of trivial representations, the reader can consider the Kummer surface \( S \); it has 16 independent Hodge cycles in \( H^2(S) \), which in turn will lead to trivial representations. More generally, we notice that the Jordan totient function \( J_1(n + 1) = O(n^4) \) governs the number of trivial representations for \( Kummer \). The exceptional O'Grady’s 10-dimensional example is surprisingly easier to handle. The reason for this is that the LLV algebra \( so(4, 22) \) is large compared to the Euler number \( e(X) \). Once one knows that the odd cohomology vanishes, there is not much space remaining for the complement of the Verbitsky component in the LLV decomposition of \( H^*(X) \). On the other hand, O’Grady’s 6-dimensional example is harder, as there are two combinatorial solutions matching the Hodge diamond of \[ MRS18 \]. In order to find the right choice for the LLV representation in the OG6 case, we need to revisit the geometric construction of OG6 used in \[ Rap07 \] and \[ MRS18 \].

The second part of the paper is concerned with Nagai’s conjecture (1.9). First, in Section 4, we discuss the relationship between higher period maps and the LLV algebra. Then, our main representation theoretic criterion (Proposition 1.11) is established in Section 5. Using our computation of the LLV decomposition for the known cases (Theorem 1.1), we prove Theorem 1.14 in Section 6. As noted, this is a stronger version of Theorem 1.10, concluding the proof of Nagai’s conjecture for all known examples of hyper-Kähler manifolds.

For reader’s convenience, we briefly review some relevant representation theory facts in the Appendix.

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While preparing this manuscript, the papers \[ Sol18 \] and \[ Sol19 \] appeared. We acknowledge their influence; in particular, occasionally they helped streamline some of our arguments.

2. The Looijenga–Lunts–Verbitsky algebra for hyper-Kähler manifolds

A compact hyper-Kähler manifold (aka irreducible holomorphic symplectic manifold) \( X \) is a simply connected Kähler manifold such that \( H^0(X, \Omega^2_X) \) is generated by a global holomorphic symplectic 2-form \( \sigma \). To fix the notation, \( X \) will denote a compact hyper-Kähler manifold (not necessarily projective) of dimension \( 2n \). Throughout the paper, we will use

\[
\tilde{V} = H^2(X, \mathbb{Q}), \quad \tilde{q} : \tilde{V} \to \mathbb{Q}
\]

for the second cohomology endowed with the (rational) Beauville-Bogomolov quadratic form \( \tilde{q} \) of \( X \) (i.e. \( \tilde{q}(\alpha)^n = c \cdot (\alpha)^{2n} \), for some constant \( c \)). Note that the quadratic space \( (\tilde{V}, \tilde{q}) \) (and the associated Lie algebra \( \mathfrak{so}(\tilde{V}, \tilde{q}) \)) are diffeomorphism invariants of \( X \). The purpose of this section is to introduce, following Verbitsky \[ Ver95 \] and Looijenga–Lunts \[ LL97 \], the Looijenga–Lunts–Verbitsky (LLV) algebra \( \mathfrak{g} \), which enhances \( \mathfrak{so}(\tilde{V}, \tilde{q}) \). The LLV algebra \( \mathfrak{g} \) acts naturally on the cohomology algebra \( H^*(X) \), giving raise to a more refined diffeomorphism invariant of \( X \), the LLV decomposition of the cohomology. After reviewing the basic structure and properties of \( \mathfrak{g} \) and its action on cohomology, we discuss the interplay between \( \mathfrak{g} \) and the natural Hodge structure on \( H^*(X) \) (assuming a complex structure on \( X \) was fixed) and the associated Mumford-Tate algebra \( \tilde{m} \) (an analytic invariant).

2.1. Looijenga–Lunts–Verbitsky algebra. The LLV algebra \( \mathfrak{g} \) and the associated LLV decomposition of \( H^*(X) \) generalize the usual hard Lefschetz \( \mathfrak{sl}(2) \)-decomposition of the cohomology in the presence of a Kähler class \( \omega \) on \( X \) (for the moment \( X \) can be any Kähler manifold). Specifically, recall that \( \omega \) defines two operators, the Lefschetz operator \( L_{\omega} = \omega \cup - \), and the inverse Lefschetz operator \( \Lambda_{\omega}(= *^{-1}L_{\omega}*) \). Then,
$L_\omega$ and $\Lambda_\omega$ generate an $\mathfrak{sl}(2) \subset \mathfrak{gl}(H^*(X))$ acting on $H^*(X)$. Hard Lefschetz is equivalent to the resulting $\mathfrak{sl}(2)$-decomposition of the cohomology. Looijenga–Lunts [LL97] have formalized this process and avoided the use of the Hodge star operator $\ast$. To start, note that
\[ [L_\omega, \Lambda_\omega] = h, \]
where $h$ is the degree operator
\[ h : H^*(X, \mathbb{Q}) \rightarrow H^*(Y, \mathbb{Q}), \]
\[ \eta \mapsto (k - \dim X)\eta \quad \text{for} \quad \eta \in H^k(X, \mathbb{Q}). \]

It follows that $\{L_\omega, h, \Lambda_\omega\}$ is an $\mathfrak{sl}(2)$-triple. The operator $L_\eta = \eta \cup -$ is well defined for any cohomology class $\eta \in H^2(X)$, while $h$ is independent of any choice. The key observation now is that, due to the Jacobson–Morozov Theorem, the existence of an operator (automatically unique) $\Lambda_\eta$ completing $\{L_\eta, h\}$ to an $\mathfrak{sl}(2)$-triple is an open algebraic condition on the classes $\eta \in H^2(X)$. Thus, the dual Lefschetz operator $\Lambda_\eta$ can be defined for almost all classes $\eta$, independent of being a Kähler class, or even of the complex structure of $X$. This allows to define a Lie algebra $\mathfrak{g}$ (clearly definable over $\mathbb{Q}$) containing all these operators. By construction, $\mathfrak{g}$ is a diffeomorphism invariant of $X$, $\mathfrak{g}$ acts on $H^*(X)$, and any $\mathfrak{sl}(2)$ Lefschetz decomposition factors through $\mathfrak{g}$. The Kähler assumption is needed only to conclude that the set of $\eta \in H^2(X)$ for which $\Lambda_\eta$ is defined is a non-empty (and thus dense) open Zariski set.

**Definition 2.2** ([LL97]). Let $X$ be a compact Kähler manifold (not necessarily hyper-Kähler). The Looijenga–Lunts–Verbitsky (LLV) algebra\(^2\) $\mathfrak{g}(X)$ of $X$ is the Lie subalgebra of $\mathfrak{gl}(H^*(X, \mathbb{Q}))$ generated by all formal Lefschetz and dual Lefschetz operators $L_\eta, \Lambda_\eta \in \mathfrak{gl}(H^*(X, \mathbb{Q}))$ associated to almost all elements $\eta \in H^2(X, \mathbb{Q})$.

The LLV algebra $\mathfrak{g}(X)$ is a semisimple Lie algebra defined over $\mathbb{Q}$ (cf. [LL97, (1.9)]). We are interested in its structure and action on cohomology when $X$ is a compact hyper-Kähler manifold (which we assume from now on). For notational simplicity, we write
\[ \mathfrak{g} = \mathfrak{g}(X) \]
if no confusion is likely occur.

2.1.1. *The structure of the LLV algebra (over $\mathbb{Q}$) for hyper-Kähler manifolds.* The semisimple degree operator $h \in \mathfrak{g}$ induces an eigenspace decomposition of $\mathfrak{g}$. In the case of hyper-Kähler manifolds, only degrees 2, 0, and $-2$ occur\(^3\) and thus we have an eigenspace decomposition
\[ \mathfrak{g} = \mathfrak{g}_2 \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{-2} \] (2.3)
with respect to $h$ acting on $\mathfrak{g}$ by the adjoint action. The 0-eigenspace $\mathfrak{g}_0$ is a reductive subalgebra of $\mathfrak{g}$, which can be then decomposed as
\[ \mathfrak{g}_0 = \mathfrak{g} \oplus \mathbb{Q} \cdot h, \]
where $\mathfrak{g}$ is the semisimple part ($\mathfrak{g} = [\mathfrak{g}_0, \mathfrak{g}_0]$), and the 1-dimensional center $\mathfrak{z}(\mathfrak{g}_0)$ is spanned by the degree operator $h$. We refer to $\mathfrak{g}$ as the reduced LLV algebra of $X$. Since $\mathfrak{g} \subseteq \mathfrak{g}_0$ consists of degree 0 operators, the induced $\mathfrak{g}$-action on $H^*(X)$ preserves the degree. That is, we have a representation
\[ \rho_h : \mathfrak{g} \rightarrow \text{End}(H^k(X, \mathbb{Q})). \]
(2.5)

In particular, $\mathfrak{g}$ acts on $H^2(X)$. On the other hand, it preserves the cup product as a derivation:
\[ e.(x \cup y) = (e.x) \cup y + x \cup (e.y) \quad \text{for} \quad e \in \mathfrak{g}, \quad x, y \in H^*(X, \mathbb{Q}). \]
(2.6)

It follows that
\[ \mathfrak{g} \subset \mathfrak{so}(\bar{V}, \bar{q}), \]
where as before $\bar{V} = H^2(X, \mathbb{Q})$ endowed with the Beauville–Bogomolov form $\bar{q}$. In fact, the equality holds, and $\bar{V} = H^2(X)$ is the standard representation of $\mathfrak{g}$. More precisely, $\mathfrak{g}$ and $\mathfrak{g}$ can be described as follows:

\(^2\) $\mathfrak{g}(X)$ is called the total Lie algebra of $X$ in [LL97], and denoted by $\mathfrak{g}_{\text{tot}}(X)$.

\(^3\) In general, the Lefschetz operators $L_\eta$ commute, but the dual Lefschetz operators $\Lambda_\eta$ do not. For hyper-Kähler manifolds and abelian varieties, $\Lambda_\eta$ do commute, resulting in the restricted range of weights.
Theorem 2.7 (Looijenga–Lunts, Verbitsky). Let $X$ be a compact hyper-Kähler manifold. Then the LLV and reduced LLV algebras of $X$ are described by

\[ \mathfrak{g} \cong \mathfrak{so}(\bar{V}, \bar{q}), \]
\[ \mathfrak{g} \cong \mathfrak{so}(\bar{V} \oplus \mathbb{Q}^2, \bar{q} \oplus (1)_{\mathbb{Q}}) \]

In particular,

\[ \mathfrak{g}_R \cong \mathfrak{so}(3, b_2(X) - 3) \quad \text{and} \quad \mathfrak{g}_R \cong \mathfrak{so}(4, b_2(X) - 2). \]

Proof. The isomorphism over $\mathbb{R}$ is [LL97, Prop. 4.5] (also [Ver95]). The isomorphism over $\mathbb{Q}$ is not clearly addressed in the literature, so we provide the details here. For the reduced LLV algebra $\bar{g}$, since $\mathfrak{g} \subset \mathfrak{so}(\bar{V}, \bar{q})$ (both defined over $\mathbb{Q}$), the equality follows by dimension reasons.

For the identification of $\mathfrak{g}$ we use the description from [KSV19, Lemma 3.9] of $\mathfrak{g}$ in terms of the subalgebra $\bar{g}$. Starting from the decomposition

\[ \mathfrak{g} = \mathfrak{g}_{-2} \oplus (\bar{\mathfrak{g}} \oplus \mathbb{Q}h) \oplus \mathfrak{g}_2, \]

one sees that $\bar{g} \cong \mathfrak{so}(\bar{V}, \bar{q}) \cong \wedge^2 \bar{V}$, and $\mathfrak{g}_{\pm 2} \cong \bar{V}$ as $\bar{g}$-representations (e.g. $\mathfrak{g}_2$ is generated by commuting Lefschetz operators $L_\eta$ for $\eta \in H^2(X) = \bar{V}$). Identifying $\mathfrak{g}_{\pm 2}$ with $\bar{V}$, and $\bar{g}$ with $\wedge^2 \bar{V}$ by the rule

\[ a \wedge b \mapsto \frac{1}{2} (\bar{q}(a, -) \otimes b - \bar{q}(b, -) \otimes a) \]

(recall $a, b \in \bar{V} = H^2(X, \mathbb{Q})$), we have the following bracket rules (which determine $\mathfrak{g}$ starting from $\bar{g}$):

1. The obvious grading relations:
   \[ [h, a] = -2a, \quad [h, b] = 2b, \quad [h, e] = 0 \quad \text{for} \quad a \in \mathfrak{g}_{-2}, \ b \in \mathfrak{g}_2, \ e \in \bar{g}; \]
   \[ [a, a'] = 0 \quad \text{for} \quad a, a' \in \mathfrak{g}_{-2}. \]
   \[ [b, b'] = 0 \quad \text{for} \quad b, b' \in \mathfrak{g}_2. \]

2. The identifications $\mathfrak{g} = \wedge^2 \bar{V} \text{ and } \mathfrak{g}_{\pm 2} = \bar{V}$ as $\bar{g}$-representations, i.e.
   \[ [e, e'] \quad \text{for} \quad e, e' \in \bar{g} \text{ is defined by the Lie bracket operation on } \bar{g}; \]
   \[ [e, a] = e.a \in \mathfrak{g}_{-2}, \quad [e, b] = e.b \in \mathfrak{g}_2 \text{ for } a \in \mathfrak{g}_{-2}, \ b \in \mathfrak{g}_2, \ e \in \bar{g}. \]

3. Finally, the interesting cross-term relation:
   \[ [a, b] = a \wedge b + \bar{q}(a, b)h \in \mathfrak{g}_0 \quad \text{for} \quad a \in \mathfrak{g}_{-2}, \ b \in \mathfrak{g}_2. \]

All of these bracket relations are defined over $\mathbb{Q}$. On the other hand, we note that the bracket relations above are exactly the same as those for $\mathfrak{so}(\bar{V} \oplus \mathbb{Q}^2, \bar{q} \oplus (1)_{\mathbb{Q}})$ described in terms of $\mathfrak{so}(\bar{V}, \bar{q})$. (Recall, in particular, that as $\mathfrak{so}(\bar{V}, \bar{q})$ representation it holds $\wedge^2 (\bar{V} \oplus \mathbb{Q}^2) = \bar{V} \oplus (\wedge^2 \bar{V} \oplus \mathbb{Q}) \oplus \bar{V}.$) \hfill \Box

Example 2.11. If $X$ is a $K3$ surface, $H^*(X) \text{ is naturally endowed with the Mukai pairing } H^*(X, \mathbb{Q}) \cong H^2(X, \mathbb{Q}) \oplus U$, even defined over $\mathbb{Z}$. In terms of representations, $H^2(X, \mathbb{Q})$ is the standard representation of $\bar{g}$ (whose real form is $\mathfrak{so}(3, 19)$), and $H^*(X, \mathbb{Q})$ is the standard representation of $\mathfrak{g}$ (whose real form is $\mathfrak{so}(4, 20)$). More interesting is the case of Kummer surfaces, which can be considered as the case $n = 1$ in the series $\text{Kum}_n$. A Kummer surface is a $K3$ surface, but by construction it contains 16 independent ($-2$)-curves (all it matters here is that for a general non-projective Kummer $X$, $\dim_{\mathbb{Q}} (H^{1,1}(X) \cap H^2(X, \mathbb{Q})) = 16$). The LLV algebra $\mathfrak{g}$ in this case has the real form $\mathfrak{so}(4, 4)$ and the associated decomposition of the cohomology is $H^*(X, \mathbb{Q}) = V \oplus \mathbb{Q}^{16}$ with $V$ the standard $\mathfrak{g}$-representation.

Motivated by Theorem 2.7 above, we would rather like to consider the Mukai completion

\[ V = \bar{V} \oplus \mathbb{Q}^2, \quad q = \bar{q} \oplus (1)_{\mathbb{Q}} \]

of $\bar{V} = H^2(X, \mathbb{Q})$ as a more natural object associated to the cohomology of $X$. Theorem 2.7 says that $\bar{V}$ is the standard representation of the reduced LLV algebra $\bar{g}$, while the Mukai completion $V$ is the standard representation of $\mathfrak{g}$ (N.B. only for K3 surfaces, $H^*(X, \mathbb{Q}) \cong V$).

Corollary 2.12. Let $X$ be a hyper-Kähler manifold, and $r = [b_2(X)/2]$. The LLV algebra $\mathfrak{g}$ is a simple Lie algebra of type $B_{r+1}$ or $D_{r+1}$, depending on the parity of $b_2(X)$. Its reduced form $\bar{g}$ is a simple Lie algebra of type $B_r$ or $D_r$. \hfill \Box
2.1.2. The LLV decomposition. The Looijenga–Lunts–Verbitsky algebra \( \mathfrak{g} \) is by definition a subalgebra of \( \mathfrak{gl}(H^*(X, \mathbb{Q})) \) – it acts on the full cohomology \( H^*(X, \mathbb{Q}) \). Since \( \mathfrak{g} \) consists of only even degree operators (2.3), this action preserves the even and odd cohomology; that is, the action of \( \mathfrak{g} \) preserves the direct sum

\[
H^*(X, \mathbb{Q}) = H^*_\text{even}(X, \mathbb{Q}) \oplus H^*_\text{odd}(X, \mathbb{Q}).
\]

Since \( \mathfrak{g} \) is semisimple, the decomposition (2.13) may be further refined. We have

\[
H^*(X, \mathbb{Q}) = \bigoplus \mu V^\otimes \mu,
\]

with \( V_\mu \) the irreducible \( \mathfrak{g} \)-module of highest weight \( \mu \). We call (2.14) the LLV decomposition; it is a basic diffeomorphism invariant of \( X \).

With the notation of Appendix A, we write \( \mu = (\mu_0, \ldots, \mu_r) \) to indicate that \( \mu = \sum_i \mu_i \varepsilon_i \). (Here \( \varepsilon \) are weights of the standard representation \( V \).) For example, \( V_\mathfrak{g} \) is the “largest” irreducible subrepresentation of \( \text{Sym}^n V \).

**Theorem 2.15** (Verbitsky). Let \( X \) be a compact hyper-Kähler manifold \( X \) of dimension \( 2n \). Then the subalgebra \( \text{SH}^2(X, \mathbb{Q}) \subset H^*(X, \mathbb{Q}) \) generated by \( H^2(X, \mathbb{Q}) \) is an irreducible \( \mathfrak{g} \)-module \( V_\mathfrak{g} \subset \text{Sym}^n V \) of highest weight \( \mu = (n) \).

**Proof.** By definition \( \text{SH}^2(X, \mathbb{Q}) \) is the subalgebra generated by the second cohomology. Hence, every element in this subalgebra can be expressed by a linear combination of the product \( x_1 \cdots x_k \) of elements in the second cohomology \( x_i \in H^2(X, \mathbb{Q}) \). From (2.6), one directly sees that \( \text{SH}^2(X, \mathbb{Q}) \) is \( \mathfrak{g} \)-invariant. Let us further show that it is in fact invariant under the full \( \mathfrak{g} \)-action.

Recall the decomposition \( \mathfrak{g} = \mathfrak{g}_{\mathfrak{h}} \oplus (\mathfrak{g} \oplus \mathfrak{Q}h) \oplus \mathfrak{g}_2 \) in (2.3). The algebra \( \text{SH}^2(X, \mathbb{Q}) \) is clearly \( \mathfrak{h} \)-invariant. Thus it is enough to show \( \text{SH}^2(X, \mathbb{Q}) \) is closed under the \( \mathfrak{g}_2 \)-action and \( \mathfrak{g}_{\mathfrak{h}} \)-action. Any element in \( \mathfrak{g}_2 \) is of the form \( L_x \) for \( x \in H^2(X, \mathbb{Q}) \), the multiplication operator by \( x \). Hence \( \text{SH}^2(X, \mathbb{Q}) \) is closed under \( L_x \) by definition. The vector space \( \mathfrak{g}_{\mathfrak{h}} \) is generated by the operators \( \Lambda_x \) for \( x \in H^2(X, \mathbb{Q}) \). To prove \( \text{SH}^2(X, \mathbb{Q}) \) is closed under \( \Lambda_x \), we need the following standard trick in representation theory. For any \( x_1, \ldots, x_k \in H^2(X, \mathbb{Q}) \), we have

\[
\Lambda_x(x_1x_2 \cdots x_k) = \Lambda_x(L_{x_1}(x_2 \cdots x_k)) = |L_{x_1}, \Lambda_x|((x_2 \cdots x_k)) - L_{x_1}(\Lambda_x(x_2 \cdots x_k)).
\]

Since \( [L_{x_1}, \Lambda_x] \in \mathfrak{g}_0 = \mathfrak{g} \oplus \mathfrak{Q}h \), we know the first component is contained in \( \text{SH}^2(X, \mathbb{Q}) \). Hence to prove \( \Lambda_x(x_1x_2 \cdots x_k) \in \text{SH}^2(X, \mathbb{Q}) \), it is enough to prove \( \Lambda_x(x_2 \cdots x_k) \) is contained in \( \text{SH}^2(X, \mathbb{Q}) \). Now use the induction on \( k \). This proves \( \text{SH}^2(X, \mathbb{Q}) \) is closed under \( \Lambda_x \), and hence closed under the full \( \mathfrak{g} \)-action.

By restricting to \( \mathfrak{g} \subset \mathfrak{g} \), we may regard \( \text{SH}^2(X, \mathbb{Q}) \) as a \( \mathfrak{g} \)-representation. From that perspective, Verbitsky [Ver96] showed that

\[
\text{SH}^2(X, \mathbb{Q}) \cong \text{Sym}^n \mathfrak{V} \oplus \text{Sym}^{n-1} \mathfrak{V} \oplus \cdots \oplus \mathfrak{V} \oplus \mathfrak{Q} \oplus \mathfrak{Q}^2
\]
as a \( \mathfrak{g} \)-module. Then the branching rules for \( \mathfrak{g} \subset \mathfrak{g} \) (§B.2) force \( \text{SH}^2(X, \mathbb{Q}) = V_\mathfrak{g} \) as a \( \mathfrak{g} \)-representation. \( \square \)

**Remark 2.17.** Bogomolov [Bog96] showed that

\[
\text{S}(H^2(X, \mathbb{Q})) \cong \text{Sym}^{n+1} H^2(X, \mathbb{Q})/(x^{n+1} : x \in H^2(X, \mathbb{Q}), \quad \bar{q}(x) = 0)
\]
as algebras.

**Definition 2.19.** We call \( \text{S}(H^2(X, \mathbb{Q})) \cong V_\mathfrak{g} \) the Verbitsky component of \( H(X, \mathbb{Q}) \).

Since \( \mathfrak{g} \) is semisimple, the cohomology admits a \( \mathfrak{g} \)-module decomposition

\[
H^*(X, \mathbb{Q}) = V_\mathfrak{g} \oplus V'.
\]

One of our goals in the paper is to describe the complement \( V' \) for the known cases of compact hyper-Kähler manifolds \( X \) (cf. Section 3). For arbitrary hyper-Kähler manifolds \( X \) we will see that the multiplicity of \( V_\mathfrak{g} \) in \( H^*(X, \mathbb{Q}) \) is one (Proposition 2.32); equivalently, \( V' \) does not contain an irreducible \( \mathfrak{g} \)-module of highest weight \( \mu = (n) \).
Remark 2.21. As the proof of Theorem 2.15 indicates it is sometimes convenient to restrict the $\mathfrak{g}$-action on $H^*(X)$ to a $\bar{\mathfrak{g}}$-action (2.5), and apply branching rules. This argument will reappear again throughout the paper. Often, this restricted action is easier to understand. However, one of our main conclusions here is that it is better to consider action of the larger $\mathfrak{g} \supset \bar{\mathfrak{g}}$. Essentially because the larger algebra encodes more symmetries; as a $\mathfrak{g}$-module, the cohomology admits fewer irreducible subrepresentations.

2.2. Further decompositions of the cohomology. We now discuss some finer decompositions of the cohomology which are obtained once certain choices have been made. For instance, the choice of complex structure determines a Hodge structure on $H^*(X)$ (which can be regarded as a decomposition with respect to the Deligne torus $\mathbb{S}$). Similarly, the choice of a twistor family (or equivalently a hyper-Kähler metric) determines an $\mathfrak{so}(4,1)$-decomposition of the cohomology $H^*(X)$, originally discovered by Verbitsky [Ver90, Ver95]. Either of these finer decompositions factor through the LLV algebra, and in a certain sense the LLV algebra is the smallest subalgebra of $\mathfrak{gl}(H^*(X, \mathbb{R}))$ containing all these decompositions. More precisely, the LLV algebra $\mathfrak{g}$ is generated by the (generic) Mumford–Tate algebra $\mathfrak{m}$ (see §2.3) and Verbitsky’s algebra $\mathfrak{so}(4,1)$ (see §2.2.2). Furthermore, the LLV algebra has the advantage of being defined over $\mathbb{Q}$.

2.2.1. Complex structures and Looijenga–Lunts–Verbitsky algebra. So far, the complex structure on $X$ was not used in our discussion. In this subsection, we would like to take the complex structure into account and understand how it interacts with the LLV algebra $\mathfrak{g}$. Verbitsky’s Global Torelli for compact hyper-Kähler manifolds implies that the complex structure on $X$ is captured by the Hodge structure on the cohomology $H^*(X, \mathbb{Q})$, and in fact $H^2(X, \mathbb{Q})$, up to some finite ambiguity.

Given a 2n-dimensional hyper-Kähler $X$, we have a degree operator $h \in \mathfrak{g}_0 \subset \mathfrak{g}$

$$h : H^*(X, \mathbb{Q}) \to H^*(X, \mathbb{Q}), \quad x \mapsto (k - 2n)x \quad \text{for} \quad x \in H^k(X, \mathbb{Q}). \quad (2.22)$$

Assuming a complex structure on $X$ was fixed, we obtain a second operator $f \in \mathfrak{gl}(H^*(X, \mathbb{R}))$ defined by

$$f : H^*(X, \mathbb{R}) \to H^*(X, \mathbb{R}), \quad x \mapsto (q - p)\sqrt{-1}x \quad \text{for} \quad x \in H^{p,q}(X), \quad (2.23)$$

capturing the Hodge structure of the cohomology. While $h$ and $\mathfrak{g}$ are defined over $\mathbb{Q}$, $f$ is in general only defined over $\mathbb{R}$. One sees that $f \in \mathfrak{g}_\mathbb{R} \subset \mathfrak{gl}(H^*(X, \mathbb{R}))$, and, in fact, a stronger statement holds.

Proposition 2.24. The operator $f \in \mathfrak{gl}(H^*(X, \mathbb{R}))$ in (2.23) is contained in $\bar{\mathfrak{g}}_\mathbb{R}$ as a semisimple element.

Proof. Fix a hyper-Kähler metric $g$ on $X$ inducing the twistor complex structures $I, J, K$ with $I$ being the original complex structure of $X$. We can associate to the complex structures $I, J, K$ the dual Lefschetz operators $L_I, L_J, L_K$ the dual Lefschetz operators associated to them.

Let $f \in \mathfrak{gl}(H^*(X, \mathbb{R}))$ be the Hodge operator as in (2.23). Verbitsky [Ver90] showed that

$$f = -[L_J, \Lambda_K] = -[L_K, \Lambda_J] \quad \text{on} \quad H^*(X, \mathbb{R}).$$

Thus, $f$ is contained in $\mathfrak{g}_\mathbb{R}$ by Definition 2.2. Since $f$ is a degree 0 operator, we have in fact $f \in \mathfrak{g}_{0,\mathbb{R}}$. One can similarly define the operators $f_J, f_K \in \mathfrak{g}_{0,\mathbb{R}}$ for the Hodge structures of other complex structures $J$ and $K$, with Verbitsky’s relations $f_J = -[L_K, \Lambda_J]$ and $f_K = -[L_I, \Lambda_J]$. By symmetry, we get $f_J, f_K \in \mathfrak{g}_{0,\mathbb{R}}$. Now using the Jacobi identities and the relations above, we get

$$[f_J, f_K] = [[L_K, \Lambda_I], [L_I, \Lambda_J]]$$

$$= [L_K, [\Lambda_I, [L_I, \Lambda_J]]] - [\Lambda_I, [L_K, [L_I, \Lambda_J]]]$$

$$= [L_K, [\Lambda_I, L_J]] + [L_K, [L_I, \Lambda_J]] - [\Lambda_I, [L_K, L_I, \Lambda_J]] - [\Lambda_I, [L_I, L_K, \Lambda_J]]$$

$$= [L_K, [-h, \Lambda_J]] + [L_K, [L_I, 0]] - [\Lambda_I, [0, \Lambda_J]] - [\Lambda_I, [L_I, -f]]$$


We conclude $f \in \mathfrak{g}_{0,\mathbb{R}} \subset \bar{\mathfrak{g}}_\mathbb{R}$. Finally, $f$ is a semisimple element of $\bar{\mathfrak{g}}_\mathbb{R}$ since $f$ acts diagonalizably on the faithful $\bar{\mathfrak{g}}_\mathbb{R}$-representation $H^2(X, \mathbb{Q})$. \qed
Now we have two operators \( h \in \mathfrak{g} \) and \( f \in \mathfrak{g}_R \). The action \( h \in \mathfrak{g} \) on the standard \( \mathfrak{g} \)-module \( V \) induces an \( h \)-eigenspace decomposition
\[
V = V_{-2} \oplus V_0 \oplus V_2, \quad \dim V_{\pm 2} = 1, \ V_0 = \tilde{V}.
\] (2.25)
Here the lower indexes indicates the eigenvalues of \( h \). Similarly, the action \( f \in \mathfrak{g}_C \) on \( V_C \) induces a \( f \)-eigenspace decomposition
\[
V_C = V_{C,-2\sqrt{-1}} \oplus V_{C,0} \oplus V_{C,2\sqrt{-1}}, \quad \dim V_{C,\pm 2\sqrt{-1}} = 1.
\] (2.26)
Since \( h, f \in \mathfrak{g}_C \) are commuting semisimple elements, there exists a Cartan subalgebra \( \mathfrak{h} \subset \mathfrak{g}_C \) containing both \( h \) and \( f \). Recall that \( \mathfrak{g} \) is a simple Lie algebra of rank \( r + 1 \), so its Cartan subalgebra \( \mathfrak{h} \) has dimension \( r + 1 \). We will use the notation \( \varepsilon_0, \ldots , \varepsilon_r \) to denote our preferred choice of a basis of \( \mathfrak{h} \) in Appendix A. Note that we start the index from 0. Now the \( h \) and \( f \)-eigenspace decompositions above have the following interpretation. This was already appeared in the discussion of [Sol18, §3.4]

**Lemma 2.27.** Let \( \mathfrak{h} \subset \mathfrak{g}_C \) be a Cartan subalgebra containing both \( h \) and \( f \). Then we must have
\[
h = \pm \varepsilon_i^\vee, \quad \sqrt{-1} f = \pm \varepsilon_j^\vee \quad \text{for some } i \neq j.
\]

**Proof.** The idea here is essentially the same as in Deligne’s approach to the classification of Hermitian symmetric domains (see, e.g., Milne’s note [Mil11, p.12]). By definition, the weights \( \varepsilon_0, \cdots, \varepsilon_r \) are obtained by the weight decomposition of the standard \( \mathfrak{g} \)-module \( V \) (see Appendix A). More specifically, we have a weight decomposition with respect to the chosen Cartan subalgebra \( \mathfrak{h} \subset \mathfrak{g}_C \)
\[
V_C = V(\pm \varepsilon_0) \oplus \cdots \oplus V(\pm \varepsilon_r) \quad \text{or} \quad V(0) \oplus V(\pm \varepsilon_0) \oplus \cdots \oplus V(\pm \varepsilon_r),
\]
where \( V(\theta) \) denotes the weight \( \theta \) subspace of \( V_C \). As an element in \( \mathfrak{h} \), \( h \) acts on the weight space \( V(\theta) \) by multiplication \( \langle \theta, h \rangle \). Now by (2.25), this implies \( \langle \theta, h \rangle = 0, -2, 2 \) for \( \theta = \pm \varepsilon_0, \cdots, \pm \varepsilon_r \) and there is only one \( \varepsilon_i \) with \( \langle \varepsilon_i, h \rangle = \pm 2 \). This forces \( h = \pm \varepsilon_i^\vee \) for some \( i = 0, \cdots , r \).

Same idea applies to \( f \), but this time we need a coefficient \( \sqrt{-1} \) as the eigenvalues of \( f \) are \( 0, \pm 2\sqrt{-1} \) in (2.26). Hence we deduce \( \sqrt{-1} f = \pm \varepsilon_j^\vee \) for some \( j = 0, \cdots , r \). Here \( i \) and \( j \) cannot be the same, as certainly \( h \) and \( f \) are linearly independent. \( \square \)

Thanks to this lemma, after choosing an appropriate positive Weyl chamber, we may assume
\[
h = \varepsilon_0^\vee, \quad \sqrt{-1} f = \varepsilon_1^\vee.
\] (2.28)
From now on, we fix an appropriate positive Weyl chamber so that we can use this condition freely.

Having discussed the complex structure, we may now consider the Hodge diamond of \( H^*(X) \). Note that the Hodge diamond is in fact derived from a Hodge structure, which is again captured by the action of the operators \( h = \varepsilon_0^\vee \) and \( f = \sqrt{-1} \varepsilon_1^\vee \). An interesting conclusion is that *any* \( \mathfrak{g} \)-module, or \( \mathfrak{g} \)-module if we ignore the weight, possesses its own Hodge structure and hence its own Hodge diamond. Let us elaborate this fact a bit more precisely.

Consider the weight decomposition of a \( \mathfrak{g} \)-module \( V_{\mu,\mathcal{C}} \). It is of the form
\[
V_{\mu,\mathcal{C}} = \bigoplus_{\theta} V_{\mu}(\theta),
\]
where \( V_{\mu}(\theta) \) denotes the weight \( \theta \) vector subspace of \( V_{\mu,\mathcal{C}} \). The Hodge decomposition is obtained by the \( (h, f) \)-eigenspace decomposition. Namely, the Hodge decomposition of \( V_{\mu} \) is
\[
V_{\mu,\mathcal{C}} = \bigoplus_{p,q} V_{\mu}^{p,q},
\] (2.29)
where \( V_{\mu}^{p,q} \) is the \( (h, f) \)-eigenspace on which \( h \) acts by multiplication \( p + q - 2n \) and \( f \) acts by multiplication \( \sqrt{-1}(q - p) \). This Hodge decomposition of \( V_{\mu,\mathcal{C}} \) can be easily deduced from the weight decomposition above. The operator \( h \) and \( f \) acts on the weight subspace \( V_{\mu}(\theta) \) by the multiplication \( \langle \theta, h \rangle \) and \( \langle \theta, f \rangle \), respectively. Hence the Hodge \((p, q)\)-component \( V_{\mu}^{p,q} \) is just a direct sum of weight subspaces \( V_{\mu}(\theta) \) with \( \langle \theta, h \rangle = p + q - 2n \) and \( \langle \theta, f \rangle = \sqrt{-1}(q - p) \).

Recalling (2.28), if we denote the weight by \( \theta = \theta_0 \varepsilon_0 + \cdots + \theta_r \varepsilon_r \), then we have
\[
\langle \theta, h \rangle = \langle \theta, \varepsilon_0^\vee \rangle = 2\theta_0, \quad \langle \theta, f \rangle = \frac{1}{\sqrt{-1}} \langle \theta, \varepsilon_1^\vee \rangle = -2\sqrt{-1}\theta_1.
\] (2.30)
This expresses $p$ and $q$ in terms of $\theta_0$ and $\theta_1$:
\[ p = \theta_0 + \theta_1 + n, \quad q = \theta_0 - \theta_1 + n. \] (2.31)

Since $\theta_i$ are always mutually integers or half-integers (see (A.2) and (A.6)), both $p$ and $q$ are integers as we expect. There are several direct consequences of this simple observation.

**Proposition 2.32.** The Hodge numbers $h^{p,q}_{(n)} = \dim V^{p,q}_{(n)}$ of the Verbitsky component $V_{(n)} \subset H^*(X, \mathbb{Q})$ satisfy $h^{p,0}_{2p} = 1$ and $h^{p+1,0}_{2p+1} = 0$ for all $0 \leq p \leq n$. In particular, the Verbitsky component occurs with multiplicity one $(m_{(n)} = 1)$ in $H^*(X, \mathbb{Q})$.

**Proof.** Since the Verbitsky component $V_{(n)} \subset H^*(X, \mathbb{Q})$ is a $g$-submodule, it is also a sub-Hodge structure. Ignoring the notion of weight for simplicity, the Hodge decomposition of $\bar{V} = H^2(X, \mathbb{Q})$ is simply the $f$-eigenspace decomposition
\[ \bar{V}_c = \bar{V}^{1,-1} \oplus \bar{V}^{0,0} \oplus \bar{V}^{-1,1}, \]
where $\dim \bar{V}^{1,-1} = \dim \bar{V}^{-1,1} = 1$. Now using the description (2.16) of the Verbitsky component, the $\bar{g}$-module structures of each degrees of $V_{(n)}$ are $\text{Sym}^k \bar{V}$, which from the above Hodge structure on $\bar{V}$ has the outermost Hodge numbers 1. Since the boundary Hodge numbers are $h^{2k,0} = h^{0,2k} = h^{2k,2n} = h^{2n,2k} = 1$ for compact hyper-Kähler manifolds $X$, the Verbitsky component already exhausts the boundary Hodge numbers 1.

The existence of the Hodge structure on $\bar{g}$-modules also allows us to put more restrictions on the LLV components arising on the cohomology of $X$. Note that, even without the complex structure, the fact that $h = \varepsilon_0^\mu$ captures the degree of the cohomology implies every irreducible component $V_\mu \subset H^*(X, \mathbb{Q})$ satisfies
\[ \langle \mu, h \rangle = \langle \mu, \varepsilon_0^\mu \rangle = 2 \mu_0 \leq 2n. \] (2.33)

Thus, we obtain $\mu_0 \leq n$. Taking into account also the Hodge structure, or equivalently $f$, we get a stronger inequality.

**Proposition 2.34.** Every irreducible $g$-module $V_\mu$ contained in the full cohomology $H^*(X)$ satisfies either $\mu = (n)$ or $\mu_0 + \mu_1 \leq n - 1$.

**Proof.** By Proposition 2.32, the Verbitsky component $V_{(n)}$ always exhausts all the boundary Hodge numbers of $X$. Thus, if $\mu \neq (n)$ occurs as a highest weight in the LLV decomposition, then all the nonzero $(p,q)$-component arising in $V_\mu$ satisfy $1 \leq p \leq 2n - 1$. The highest $g$-module $V_\mu$ always contains the weight $\mu$. By (2.31), all nonzero $(p,q)$-component in $V_\mu$ satisfy $p = \mu_0 + \mu_1 + n$ and $q = \mu_0 - \mu_1 + n$. Hence $\mu_0 + \mu_1 = p - n \leq n - 1$, as needed.

Similarly, we obtain the following easy restriction on the possible irreducible components of the LLV decomposition on the even and odd cohomology respectively.

**Proposition 2.35.** Let $X$ be a hyper-Kähler manifold, and $g$ its LLV algebra.

(i) Every irreducible $g$-module component $V_\mu \subset H^*_\text{even}(X)$ has integer coefficients $\mu_i \in \mathbb{Z}$. Similarly, every irreducible $g$-module component $V_\mu \subset H^*_\text{odd}(X)$ has half-integer coefficients $\mu_i \in \frac{1}{2} \mathbb{Z} \setminus \mathbb{Z}$.

(ii) Every irreducible $g$-module component $V_\lambda \subset H^{2k}_\text{even}(X)$ has integer coefficients $\lambda_i \in \mathbb{Z}$, while every irreducible $V_\lambda \subset H^{2k+1}_\text{odd}(X)$ has half-integer coefficients $\lambda_i \in \frac{1}{2} \mathbb{Z} \setminus \mathbb{Z}$.

**Proof.** Applying (2.31) to the highest weight $\mu$ of $V_\mu$, we have $p = \mu_0 + \mu_1 + n$ and $q = \mu_0 - \mu_1 + n$. If we assume $V_\mu \in H^*_\text{even}(X)$, then we have an even $p + q = 2\mu_0 + 2n$. This proves $\mu_0 \in \mathbb{Z}$, and hence by (A.2) and (A.6) all the $\mu_i$ are integers. If we assume $V_\mu \in H^*_\text{odd}(X)$, then similar argument implies $\mu_0$ is a half-integer and hence all $\mu_i$ are half-integers.

For the second statement, we cannot use the the operator $h \notin \bar{g}$, so we need to go back to (2.30). From it, we have $p - q = 2\lambda_1$. If $V_\lambda \subset H^{2k}(X)$ lives in an even cohomology, then $p - q$ is even so $\lambda_1$ is integer. Hence all $\lambda_i$ are integers. Similar argument proves the case $V_\lambda \subset H^{2k+1}(X)$.

As an immediate corollary, we see that all reduced LLV modules $H^k(X)$ are faithful $\bar{g}$-modules.

**Corollary 2.36.** If $H^k(X) \neq 0$, then the map $\rho_k : \bar{g} \to \text{End}(H^k(X))$ is injective.
2. According to Deligne, a Mumford–Tate algebra is as follows. Let $W$ be a group of $\mathbb{Q}$-algebraic Lie subalgebra of $\mathfrak{gl}(H^k(X))$ generated by these six operators. It is shown in loc. cit. that $\mathfrak{g}_g \cong \mathfrak{so}(4, 1)$. By definition of Looijenga–Lunts–Verbitsky algebra, we have an inclusion

$$\mathfrak{so}(4, 1) \cong \mathfrak{g}_g \subset \mathfrak{g}_R.$$ 

Hence, the choice of hyper-Kähler metric $g$ on $M$ induces the Verbitsky algebra $\mathfrak{g}_g$. If we vary a hyper-Kähler metric $g$ on $M$, then $\mathfrak{g}_g$ moves inside of the Looijenga–Lunts–Verbitsky algebra $\mathfrak{g}_R$.

Finally, let us discuss the relationship between the Verbitsky algebra $\mathfrak{so}(4, 1)$ and Fujiki’s work [Fuj87]. Once a hyper-Kähler metric $g$ was fixed, Fujiki constructed an $\text{Sp}(1)$-action on each cohomology $H^k(X, \mathbb{R})$ by purely differential geometric methods. He studied the $\text{Sp}(1)$-representation theory on the cohomology $H^k(X, \mathbb{R})$ and as a result, obtained Hodge decomposition-type and hard Lefschetz-type theorems. In fact, the associated $\text{Sp}(1)$ decomposition essentially coincides with the decomposition associated to Verbitsky’s $\mathfrak{g}_g \cong \mathfrak{so}(4, 1)$-decomposition (and in particular, factors through the LLV decomposition). The decomposition (2.3) of the LLV algebra $\mathfrak{g}$ induces a degree decomposition for Verbitsky’s algebra $\mathfrak{g}_g \cong \mathfrak{so}(4, 1):

$$\mathfrak{g}_g = \mathfrak{g}_{g,-2} \oplus \mathfrak{g}_{g,0} \oplus \mathfrak{g}_{g,2},$$

with $\mathfrak{g}_g \cong \mathfrak{so}(3, \mathbb{R})$. Lifting the Lie algebra $\mathfrak{so}(3, \mathbb{R})$ to the level of Lie group gives us a simply connected real Lie group $\text{Spin}(3, \mathbb{R})$, which is isomorphic to $\text{Sp}(1)$ by an exceptional isomorphism (corresponding to $B_1 \cong C_1$).

2.3. The Mumford–Tate algebra. The Looijenga–Lunts–Verbitsky algebra is a diffeomorphism invariant of a compact hyper-Kähler manifold $X$. A complex structure on $X$ is encoded by the Hodge structure on the cohomology $H^k(X, \mathbb{Q})$. This Hodge structure is in turn given by a semisimple element $f \in \mathfrak{g}_R$ (Proposition 2.24). To the Hodge structure is associated a (special) Mumford–Tate group. Here, we discuss the relationship between the Mumford–Tate algebra and the Looijenga–Lunts–Verbitsky algebra.

**Definition 2.37.** Let $W$ be a $\mathbb{Q}$-Hodge structure. Define the operators $h \in \mathfrak{gl}(W)$ and $f \in \mathfrak{gl}(W)_R$ by

$$h : W \rightarrow W, \quad x \mapsto (p + q)x \quad \text{for} \quad x \in W^{p,q},$$

$$f : W_R \rightarrow W_R, \quad x \mapsto (q - p)\sqrt{-1}x \quad \text{for} \quad x \in W^{p,q},$$

as in our previous notation (2.22) and (2.23). The special Mumford–Tate algebra of $W$ is the smallest $\mathbb{Q}$-algebraic Lie subalgebra $\overline{\mathfrak{m}}(W)$ of $\mathfrak{gl}(W)$ such that $f \in \overline{\mathfrak{m}}(W)_R$. The Mumford–Tate algebra of $W$ is $\mathfrak{m}_0(W) = \overline{\mathfrak{m}}(W) \oplus \mathbb{Q}h$.

The Mumford–Tate algebra of $W$ is usually defined as the associated Lie algebra of the Mumford–Tate group of $W$. Our definition coincides with this definition by the discussion in [Zar83, §0.3.3]. The correspondence is as follows. Let $S = \text{Res}_{\mathbb{C}/\mathbb{R}} G_{m, \mathbb{C}}$ be the Deligne torus. It is a nonsplit $\mathbb{R}$-algebraic torus of rank 2. According to Deligne, a $\mathbb{Q}$-Hodge structure $W$ is a finite dimensional $\mathbb{Q}$-vector space $W$ equipped with
an appropriate $S$-module structure on $W_\mathbb{R}$ (e.g. see [Moo99]). That is, we have a morphism of $\mathbb{R}$-algebraic groups
\[ \varphi : S \to \text{GL}(W)_\mathbb{R}, \]
with $G_{m,\mathbb{R}} \subset S \to \text{GL}(W)_\mathbb{R}$ defined over $\mathbb{Q}$. By definition, the Mumford–Tate group $\text{MT}(W)$ of $W$ is the smallest $\mathbb{Q}$-algebraic subgroup of $\text{GL}(W)$ such that $\varphi(S) \subset \text{MT}(W)_\mathbb{R}$. Now take the differential of $\varphi$. We obtain a homomorphism of $\mathbb{R}$-Lie algebras
\[ \varphi_* : \mathfrak{u}(1) \oplus \mathbb{R} \to \mathfrak{gl}(W)_\mathbb{R}. \]
The images the generators of $\mathfrak{u}(1)$ and $\mathbb{R}$ are precisely $f$ and $h$ as above, giving the equivalence of the two definitions.

Returning to the hyper–Kähler geometry, we can consider the Hodge structures of degree $k$ on $H^k(X, \mathbb{Q})$, and also of the full cohomology $H^*(X, \mathbb{Q})$. We will simply write
\[ \mathfrak{m} = \overline{\mathfrak{m}}(H^*(X, \mathbb{Q})) \]
for the special Mumford-Tate algebra associated to the full cohomology of $X$. It is the $\mathbb{Q}$-algebraic Lie algebra closure of the one-dimensional real Lie algebra $\mathbb{R}f \subset \mathfrak{gl}(H^*(X, \mathbb{R}))$. There is a close relationship between $\mathfrak{m}$, the Mumford–Tate algebras of the individual cohomologies $H^k(X, \mathbb{Q})$, and the Looijenga–Lunts–Verbitsky algebra $\mathfrak{g}$ of $X$.

**Proposition 2.38.** Let $X$ be a compact hyper-Kähler manifold of dimension $2n$.

(i) There exists an inclusion $\mathfrak{m} \subset \mathfrak{g}$. Equality holds for very general $X$.

(ii) If $0 < k < 4n$ and $H^k(X, \mathbb{Q}) \neq 0$, then $\overline{\mathfrak{m}}(H^k(X, \mathbb{Q})) = \mathfrak{m}$.

In particular, while Mumford–Tate algebras vary with the choice of complex structure on $X$, they remain subalgebras of the LLV algebra. In other words, all the Mumford–Tate algebras factor through $\mathfrak{g}$ (which is a diffeomorphism invariant).

**Proof.** The first statement of (i) is a direct consequence of Proposition 2.24. By definition, the special Mumford–Tate algebra $\mathfrak{m} \subset \mathfrak{gl}(H^*(X, \mathbb{Q}))$ is the smallest Lie algebra with $f \in \mathfrak{m}_\mathbb{R}$. By Proposition 2.24, $f \in \mathfrak{g}_\mathbb{R}$. Hence $\mathfrak{m} \subset \mathfrak{g}$.

Before proving the equality assertion of (i), let us first prove (ii). The $\mathfrak{g}$-module structure (2.5) on $H^k(X, \mathbb{Q})$ is the composition
\[ \rho_k : \mathfrak{g} \subset \mathfrak{gl}(H^*(X, \mathbb{Q})) \xrightarrow{\pi_k} \mathfrak{gl}(H^k(X, \mathbb{Q})). \]
This map $\rho_k$ is injective (Corollary 2.36). The paragraph above shows that $\mathfrak{m}$ is a posteriori the $\mathbb{Q}$-algebraic Lie algebra closure of $f$ in $\mathfrak{g}_\mathbb{R}$. Note that $\pi_k(f) \in \mathfrak{gl}(H^k(X, \mathbb{R}))$ is the operator encoding the Hodge structure of $H^k(X, \mathbb{Q})$. Thus $\overline{\mathfrak{m}}(H^k(X, \mathbb{Q}))$ is the $\mathbb{Q}$-algebraic Lie algebra closure of $\pi_k(f)$ in $\mathfrak{gl}(H^k(X, \mathbb{R}))$. But we already have $\pi_k(f) \in \pi_k(\mathfrak{g})_\mathbb{R}$, so by the same reason, $\overline{\mathfrak{m}}(H^k(X, \mathbb{Q}))$ is the $\mathbb{Q}$-algebraic Lie algebra closure of $\pi_k(f)$ in $\pi_k(\mathfrak{g})_\mathbb{R}$. But since $\rho_k$ is injective, $\pi_k$ induces an isomorphism between $\mathfrak{g}$ and $\pi_k(\mathfrak{g})$. Thus $\pi_k : \mathfrak{m} \to \overline{\mathfrak{m}}(H^k(X, \mathbb{Q}))$ is an isomorphism.

It remains to prove the equality assertion of (i). It is a general fact in the theory of Mumford–Tate groups that the special Mumford–Tate group of the Hodge structure $\tilde{V} = H^2(X, \mathbb{Q})$ of K3 type is $\text{SO}(\tilde{V}, \bar{q})$ outside of the Noether-Lefschetz locus in the period domain of $\tilde{V}$ (see [GGK12]). Since compact hyper-Kähler manifolds satisfy local Torelli theorem on second cohomology (or, even more, global Torelli theorem [Ver13, Huy12]), this means $\overline{\mathfrak{m}}(H^2(X, \mathbb{Q})) \cong \text{so}(\tilde{V}, \bar{q})$ for very general $X$. Since $\overline{\mathfrak{m}}(H^2(X, \mathbb{Q})) = \mathfrak{m}$, the equality assertion follows.

By Proposition 2.38, the full Hodge structure on $H^*(X, \mathbb{Q})$ has the same degree of transcendence as the Hodge structure on $H^2(X, \mathbb{Q})$ over the special Mumford–Tate algebra $\overline{\mathfrak{m}}(H^2(X, \mathbb{Q})) = \mathfrak{m}$. This was anticipated by the Torelli principle for the second cohomology of hyper-Kähler manifolds (e.g., [Huy03, Cor. 24.5] or [Sol19]). As a byproduct of this proposition, following Zarhin [Zar83], one can classify the special Mumford–Tate algebra of projective hyper-Kähler manifolds.
In this section, we determine the LLV decomposition for all known examples of hyper-Kähler manifolds (Theorem 1.1). We begin with a review of what is known about these cohomology groups. From our perspective these results are equivalent to describing the structure of $H^*(X)$ as $\mathfrak{m} = \mathfrak{so}(3, b_2 - 4)$ module (§2.3). This structure is the restriction of an $\mathfrak{m} \cong \mathfrak{so}(4, b_2 - 3)$ module structure (as in the proof of Theorem 2.7). We then show that this second structure is in turn the restriction to $\mathfrak{m}$ of a $\mathfrak{g} \cong \mathfrak{so}(4, b_2 - 2)$ representation. This argument works for both the $K3^{[n]}$ types and the Kum$_n$ types as the necessarily initial module structure is known [GS93], even in the non-projective cases [dCM00]. For the OG6 and OG10 types, we do not have the full $\mathfrak{m}$-module structures, only the Hodge numbers [MRS18] and [dCRS19]. For OG10 this suffices, as we are dealing with a big algebra $e(X) = 176, 904$. For OG6, these considerations reduce us to two possible LLV module structures. To identify the identity $\mathfrak{g}$-representation, we need to delve deeper into the geometric construction of [MRS18]. The proof of Theorem 1.1 is presented case by case in §§3.2–3.5.

**Remark 3.1.** Since our arguments make use of the Mumford–Tate algebra $\mathfrak{m}$, it is important that the LLV algebra $\mathfrak{g}$ is defined over $\mathbb{Q}$. However, it is more convenient to work over $\mathbb{R}$; we will do so throughout (unless $\mathbb{Q}$ coefficients are needed). For instance, this allows us to write

$$\mathfrak{g} = \mathfrak{so}(4, b_2(X) - 2), \quad \bar{\mathfrak{g}} = \mathfrak{so}(3, b_2(X) - 3)$$

ignoring the rational quadratic structure. Similarly, we write $H^*(X) = H^*(X, \mathbb{R})$. (Everything is however defined over $\mathbb{Q}$, and the discussion can be easily adapted to $\mathbb{Q}$ coefficients.)

To our knowledge, very little was previously known on the LLV decomposition for the known cases. The $K3$ surface and the Kummer surface are clear (Eg. 2.11). Recall that, for any hyper-Kähler manifold $X$, the Verbitsky component $V_m$ counts force $H^*(\mathcal{K}^3) = V_2$. The only other LLV decompositions that we are aware of are the next simplest cases, Kum$_2$ (see [LL97, Ex 4.6]) and $K3^{[3]}$ (see [Mar02, Example 14]). The main result of the section is stated as Theorem 1.1 in the introduction. For reader’s convenience we state two corollaries of this result, namely the explicit LLV decompositions for hyper-Kähler manifolds of type $K3^{[n]}$ and Kum$_n$ respectively for small values of $n$.

**Corollary 3.2.** Let $\mathfrak{g} \cong \mathfrak{so}(4, 21)$ be the LLV algebra for hyper-Kähler manifolds of $K3^{[n]}$ type (with $n \geq 2$). Then, for $n \in \{2, \cdots, 7\}$, the associated LLV decomposition of the cohomology is as follows:

- $H^*(\mathcal{K}^{[2]}[3]) = V_{(2)}$
- $H^*(\mathcal{K}^{[3]}[4]) = V_{(3)} \oplus V_{(1, 1)}$
- $H^*(\mathcal{K}^{[4]}[5]) = V_{(4)} \oplus V_{(2, 1)} \oplus V_{(2)} \oplus \mathbb{R}$
- $H^*(\mathcal{K}^{[5]}[6]) = V_{(5)} \oplus V_{(3, 1)} \oplus V_{(1, 1)} \oplus V_{(1, 1)} \oplus V_{(2)} \oplus \mathbb{R}$
- $H^*(\mathcal{K}^{[6]}[7]) = V_{(6)} \oplus V_{(4, 1)} \oplus V_{(3, 1)} \oplus V_{(3, 2)} \oplus V_{(2, 1)} \oplus V_{(2)} \oplus \mathbb{R}$
- $H^*(\mathcal{K}^{[7]}[8]) = V_{(7)} \oplus V_{(3, 1)} \oplus V_{(5, 1)} \oplus V_{(1, 1)} \oplus V_{(2)} \oplus \mathbb{R}$

**Remark 3.3.** As an illustration, let us discuss the case of hyper-Kähler manifold $X$ of $K3^{[3]}$ type. The LLV algebra of $X$ is $\mathfrak{g} \cong \mathfrak{so}(4, 21)$, and the above result says $H^*(X) = V_{(3)} \oplus V_{(1, 1)}$ as $\mathfrak{g}$-modules, or equivalently

$$H^*(X) = V_{3} \oplus V_{\varpi}$$

(3.4) in terms of the fundamental weights. Further decomposing (3.4) as a module of the reduced LLV algebra $\bar{\mathfrak{g}} = \mathfrak{so}(3, 20)$ accounts for disassembling the Mukai completion. By definition, the standard $\mathfrak{g}$-module $V$ decomposes as $V = \mathbb{R}(1) \oplus \bar{V} \oplus \mathbb{R}(-1)$ when viewed as $\bar{\mathfrak{g}}$-module. Here $\mathbb{R}(\pm 1)$ indicates the degree $\mp 2$ parts of $V$; $\bar{V}$ has degree 0. The branching rules (see Appendix B) give

$$H^*(X) = (\mathbb{R}(3) \oplus \bar{V}(2) \oplus (\text{Sym}^2 \bar{V} \oplus \bar{V})(1) \oplus (\text{Sym}^3 \bar{V} \oplus \Lambda^2 \bar{V} \oplus \mathbb{R}) \oplus (\text{Sym}^2 \bar{V} \oplus \bar{V})(-1) \oplus \bar{V}(-2) \oplus \mathbb{R}(-3),$$

where $\Lambda^2 \bar{V} = \bar{V} \oplus \bar{V}$.
which the reader will notice is much more involved than (3.4). The decomposition (3.5) yields $H^0(X) = \mathbb{R}$, $H^2(X) = \bar{V}$, $H^4(X) = \text{Sym}^2 \bar{V} \oplus \bar{V}$, and so on, recovering Markman’s computation [Mar02, Ex. 14]. Finally, specializing $X$ to $X = \text{Hilb}^3(S)$ for some $K3$ surface $S$, the generic Mumford–Tate algebra $\mathfrak{m}$ of $X$ in this locus becomes slightly smaller than $\mathfrak{g}$ (see Proposition 2.38). More specifically, we have $\mathfrak{m} = \mathfrak{so}(3, 19)$ contained in $\mathfrak{g} = \mathfrak{so}(3, 20)$. Restricting the above identity further to $\mathfrak{m}$, we recover the Göttscbe–Sorger’s formula on Hodge structures [GS93], which is equivalent to the $\mathfrak{m}$-module structure.

Similarly, we have the following formulas for the low dimensional $\text{Kum}_n$ cases.

**Corollary 3.6.** Let $\mathfrak{g} \cong \mathfrak{so}(4, 5)$ be the LLV algebra for hyper-Kähler manifolds of $\text{Kum}_n$ type (with $n \geq 2$). Then, for $n \in \{2, \ldots, 5\}$, the associated LLV decomposition of the cohomology is as follows:

- $H^*(\text{Kum}_2) = V(2) \oplus \mathbb{R}^{\otimes 80} \oplus V(1, 1) \oplus \mathbb{R}^{\otimes 16} \oplus \mathbb{R}^{\otimes 240} \oplus V(1, 1, 1) \oplus \mathbb{R}^{\otimes 5}$
- $H^*(\text{Kum}_4) = V(4) \oplus V(2, 1) \oplus V(2, 1, 1) \oplus V(1, 1, 1) \oplus \mathbb{R}^{\otimes 625} \oplus V(\frac{2}{2}, 1, 1) \oplus V(\frac{3}{2}, 1, 1) \oplus V(\frac{4}{2}, 1, 1) \oplus V(\frac{5}{2}, 1, 1) \oplus V(\frac{6}{2}, 1, 1)$
- $H^*(\text{Kum}_5) = V(5) \oplus V(3, 1) \oplus V(3, 1, 1) \oplus V(2, 1, 1) \oplus V(2, 1, 1) \oplus \mathbb{R}^{\otimes 2} \oplus V(2, 1, 1, 1) \oplus V(1, 1, 1, 1) \oplus V(1, 1, 1, 1) \oplus \mathbb{R}^{\otimes 17}$

**Remark 3.7.** We do not have closed formulas for the general case $\text{Kum}_n$ and $\text{Kum}_n$, but as one can see, the cohomology of $\text{Kum}_n$ is fairly complicated. Note in particular, the presence of several spin type representations, and the large number of trivial representations. The number of trivial representations is controlled by the fourth Jordan totient function $J_4(n+1) \sim (n+1)^4$. Specifically note that in the range that we have worked out the representations explicitly ($n \in \{1, \ldots, 5\}$), the values of $J_4(n+1)$ are 15, 80, 240, 624, and 1, 200, while the number of trivial representations is 16, 80, 240, 625, and 1, 200 respectively. Geometrically, this means that a variety of $\text{Kum}_n$ type contains many Hodge cycles (of order $n^4$) even if it is non-projective.

Other consequences of Theorem 1.1 are formulas for the generating series for the Euler numbers, the Poincaré polynomials, and Hodge–Deligne polynomials for the two series $\text{Kum}_n$ and $\text{Kum}_n$. In the case of $\text{Kum}_n$, we recover the formulas of Göttscbe (see esp. [G94, Thm 2.3.10] and [G94, Rem 2.3.12]).

**Corollary 3.8.** The generating series for $\text{Kum}_n$ are as follows.

(i) The generating series for the Euler numbers of $\text{Kum}_n$ is

$$
\sum_{n=0}^{\infty} \frac{e^{(\text{Kum}_n)}}{1 - q^n} = \prod_{m=1}^{\infty} \frac{1}{(1 - q^m)^{24}} = \frac{q}{\Delta(q)},
$$

where $\Delta(q)$ is the weight 12 modular form.

(ii) The generating series for the Poincaré polynomials of $\text{Kum}_n$ is

$$
\sum_{n=0}^{\infty} b(K^3[n], t) q^n = \prod_{m=1}^{\infty} \frac{1}{(1 - t^2 q^m)(1 - t^{-2} q^m)(1 - q^m)^2},
$$

where $b(K^3[n], t)$ indicates the Poincaré polynomial with the Betti numbers $b_k(K^3[n])$ encoded in the coefficient of $t^{k-2n}$.

(iii) The generating series for the Hodge–Deligne polynomials of $\text{Kum}_n$ is

$$
\sum_{n=0}^{\infty} h(K^3[n], s, t) q^n = \prod_{m=1}^{\infty} \frac{1}{(1 - stq^m)(1 - st^{-1}q^m)(1 - s^{-1}tq^m)(1 - s^{-1}t^{-1}q^m)(1 - q^m)^{20}},
$$

where $h(K^3[n], s, t)$ indicates the Hodge–Deligne polynomial with the Hodge numbers $h^{p, q}(K^3[n])$ encoded in the coefficient of $s^{p-n} t^{q-n}$.

**Proof.** Recall the discussion in §2.2 that the Hodge structure is captured by the $\mathfrak{g}$-module structure. Specifically, the Hodge component $W^{p,q}$ of a $\mathfrak{g}$-module $W$ is the direct sum of the weight spaces $W(\theta)$, for the weights $\theta = \theta_0 \varepsilon_0 + \cdots + \theta_{11} \varepsilon_{11}$ such that

$$p = \theta_0 + \theta_1 + n, \quad q = \theta_0 - \theta_1 + n,$$
cf. (2.29) and (2.31). Note that the dimension of $W(\theta)$ is captured by the coefficient of $x_0^0 \cdots x_{11}^1$ in the formal character. Setting $x_1 = st$, $x_2 = st^{-1}$, $x_3 = \cdots = x_{11} = 1$ gives us the transformation
\[ x_0^0 \cdots x_{11}^1 = s^{0+\theta_1}t^{0-\theta_1} = s^{p-n}t^{q-n}, \]
whose coefficient contributes to the Hodge number $h^{p,q}$ of $W$. This means setting $x_0 = st$, $x_1 = st^{-1}$, $x_2 = \cdots = x_{11} = 1$ in (1.2) of Theorem 1.1 gives us the generating series of the Hodge–Deligne polynomial of this. This proves (iii).

A similar argument implies that setting $x_0 = t^2$, $x_1 = x_2 = \cdots = x_{11} = 1$ yields (ii). Finally, for (i), note that the Euler number is an alternating sum of the Betti numbers. This amounts to setting $t = -1$. \hfill \Box

For the $Kum_m$ case, specializing the generating series of Theorem 1.1(2), we obtain the following formula for the Hodge–Deligne polynomials. This formula seems new and slightly simpler than those existing in the literature, but still not as neat as in the $K3^{[n]}$ case.

**Corollary 3.9.** The generating series of Hodge–Deligne polynomials of $Kum_n$ is
\[
\sum_{n=0}^{\infty} h(Kum_n, s, t) q^n = \sum_{d=1}^{\infty} J_d(q - 1)
\]
as in (1.3), but with the formal power series $B(q)$ in this case defined by
\[
B(q) = \prod_{m=1}^{\infty} \frac{(1 + sq^m)^2(1 + s^{-1}q^m)^2(1 + tq^m)^2(1 + t^{-1}q^m)^2}{(1 - sq^m)(1 - s^{-1}q^m)(1 - tq^m)(1 - s^{-1}q^m)(1 - q^m)^2}.
\]

**Proof.** The proof is the same as that of Corollary 3.8. Setting $x_0 = st$, $x_1 = x_{11} = 1$ gives us the desired result. One can also observe the first coefficient $b_1$ of $B(q)$ is $b_1 = \frac{1}{\pi}(s + 1)^2(t + 1)^2$. \hfill \Box

### 3.1. The Mukai completion.
In this subsection, we assume $X$ to be an arbitrary compact hyper-Kähler manifold. Let $\mathfrak{m}$ and $\mathfrak{m}_0 = \mathfrak{m} \oplus \mathbb{Q}h$ be the special Mumford–Tate algebra and Mumford–Tate algebra of $X$ respectively (see Section 2). By Proposition 2.38, the Mumford–Tate algebra $\mathfrak{m}$ is contained in $\bar{\mathfrak{g}}$. If we assume $S$ is projective, then we further have a classification of the special Mumford–Tate algebra $\mathfrak{m}$ by Zarhin [Zar83]; either $\mathfrak{m} \cong so_E(T, \bar{q})$ or $u_{E_0}(T, \bar{q})$ for a totally real or CM number field $E$ (where $E$ is determined by the endomorphisms of the Hodge structure). In particular, if $E = \mathbb{Q}$, $\mathfrak{m} = so(T, \bar{q})$. The assumption $E = \mathbb{Q}$ holds when $X$ is a very general hyper-Kähler manifold.

Now, assume we had $\mathfrak{m} \cong so(T, \bar{q})$ for some sub-Hodge structure $T \subset V$. This assumption is satisfied in the following two cases:

(A) If $X$ is a very general projective hyper-Kähler manifold with a fixed polarization, then the assumption is satisfied with $T$ the transcendental Hodge structure of $V$ with $\dim T = \dim V - 1$, by the above discussion.

(B) If $X$ is a very general non-projective hyper-Kähler manifold, then the assumption is again satisfied with $T = V$, by Proposition 2.38(1).

Recall the relation between the two Lie algebras $\bar{\mathfrak{g}}$ and $\mathfrak{g}$ in Theorem 2.7. In these cases, we can formally imitate this relation to enlarge the Lie algebra $\bar{\mathfrak{g}}$ to a new Lie algebra $\mathfrak{m}$. This process is often used in the theory of moduli of sheaves on $K3$ surfaces, and called Mukai extension or Mukai completion of the second cohomology.

**Definition 3.10.** Let $(\bar{T}, \bar{q})$ be a quadratic space over $\mathbb{Q}$ and $\mathfrak{m} = so(\bar{T}, \bar{q})$ a $\mathbb{Q}$-Lie algebra. We call $(T, q) = (\bar{T} \oplus \mathbb{Q}^2, \bar{q} \oplus \left( \frac{1}{2} 1 \right))$ the Mukai completion of $(\bar{T}, \bar{q})$, and $\mathfrak{m} = so(T, q)$ the Mukai completion of $\mathfrak{m}$.

The proof of Theorem 2.7 can be interpreted that one can recover the Lie algebra $\mathfrak{g}$ as the Mukai completion of the smaller Lie algebra $\bar{\mathfrak{g}}$. Now consider the special Mumford–Tate algebra $\mathfrak{m}$ of $X$. It is contained in $\bar{\mathfrak{g}}$. If we apply the Mukai completion to $\mathfrak{m}$, then get an abstract Lie algebra $\mathfrak{m}$. Since $\mathfrak{g}$ is also the Mukai completion of $\bar{\mathfrak{g}}$, one can easily conclude
\[
\mathfrak{m} = \mathfrak{m}_{-2} \oplus \mathfrak{m}_0 \oplus \mathfrak{m}_2 \subset \mathfrak{g}, \quad \mathfrak{m}_0 = \mathfrak{m} \oplus \mathbb{Q}h, \quad \mathfrak{m}_{\pm 2} = \mathfrak{m} \cap \bar{\mathfrak{g}}_{\pm 2}.
\]

**Lemma 3.12.** Assume the special Mumford–Tate algebra $\mathfrak{m}$ of $X$ is isomorphic to $so(\bar{T}, \bar{q})$, e.g., assume $X$ satisfies either (A) or (B) above. Then its formal Mukai completion $\mathfrak{m}^{form}$ is contained in $\mathfrak{g}$, and respects the degree of $\mathfrak{g}$ in the sense of (3.11). \hfill \Box
3.2. Cohomology of Hilbert schemes of K3 surfaces. The main result of this subsection is the proof of Theorem 1.1(1) concerning the generating series for $K3^{[n]}$. Specifically, we establish:

**Theorem 3.13.** Let $\mathfrak{g}$ be the Looijenga–Lunts–Verbitsky algebra of a hyper-Kähler manifold of $K3^{[n]}$ type with $n \geq 2$. Then the generating series of the formal characters of $\mathfrak{g}$-modules $H^*(K3^{[n]})$ is

$$1 + \left( \sum_{i=0}^{11} (x_i + x_i^{-1}) \right) q + \sum_{n=2}^{\infty} \text{ch}(H^*(K3^{[n]})) q^n = \prod_{m=1}^{11} \prod_{i=0}^{n} \left( 1 - x_i q^m \right) \frac{1}{(1 - x_i^{-1} q^m)}.$$

Let $X$ be a $K3^{[n]}$ type hyper-Kähler manifold. Since the $\mathfrak{g}$-module structure on $H^*(X)$ is a diffeomorphism invariant, we may specialize $X$ to $S^{[n]}$ with $S$ a complex K3 surface. Since the statement is a diffeomorphism invariant, we may vary the complex structure of $S$. By de Cataldo–Migliorini [dCM00], this allows us to assume that the Mumford–Tate algebra of $S$ is assumed to be a polarized hyper-Kähler manifold. Since the Hodge structure of the hyper-Kähler manifold $(\bar{V}, \bar{q})$ and $(V, q)$ is the Mukai completion of the intersection pairing on the second cohomology of $S$. On the other hand, $\mathfrak{g}(X) = \mathfrak{so}(V, q)$ where $(V, q)$ is the Mukai completion of the second cohomology $(\bar{V} = H^2(X), \bar{q})$ of $X$ endowed with the Beauville–Bogomolov form. The relationship between $(V, q)$ and $H^2(S)$ is well understood. Specifically,

$$(\bar{V}, \bar{q}) = (H^2(S, \mathbb{Q}), \bar{q}_S) \oplus (-2(n-1)).$$

This implies the inclusion $\bar{\mathfrak{g}}(S) \subset \bar{\mathfrak{g}}(X)$, whence the inclusion of Looijenga–Lunts–Verbitsky algebras

$$\mathfrak{g}(S) \subset \mathfrak{g}(X).$$

The Hodge structure of the hyper-Kähler manifold $S^{[n]}$ was determined by Göttsche–Soergel [GS93]. We interpret this as giving the decomposition of $S^{[n]}$ as a representation of the Mumford–Tate algebra $\mathfrak{m} \cong \mathfrak{so}(3, 19)$. By considering the grading operator $h$, we can lift this decomposition of $H^*(S^{[n]})$ to a decomposition as a $\mathfrak{g}(S) \cong \mathfrak{so}(4, 20)$-module. Since $\mathfrak{g}(S) \cong \mathfrak{so}(4, 20)$ and $\mathfrak{g}(X) \cong \mathfrak{so}(4, 21)$ have the same rank (type $D_{12}$ and $B_{12}$ respectively), there exists a unique $\mathfrak{g}(X)$-module structure compatible (by restriction) to the $\mathfrak{g}(S)$-module structure that we have determined. We conclude that essentially formally starting from Göttsche–Soergel results, we recover the LLV decomposition for $K3^{[n]}$.

**Theorem 3.14.** Let $S$ be a K3 surface and $X = S^{[n]}$. Denote $W = H^*(S)$ by the standard $\mathfrak{g}(S)$-module. Then the $\mathfrak{g}(X)$-module structure on $H^*(X)$ is uniquely determined by the isomorphism of $\mathfrak{g}(S)$-modules

$$H^*(X) \cong \bigoplus_{\alpha+n} \left( \bigotimes_{i=1}^{n} \text{Sym}^{a_i} W \right).$$

Here $\alpha = (1^{a_1}, \cdots, n^{a_n})$ runs through all the partitions of $n = \sum_{i=1}^{n} ia_i$.

**Proof.** The main result of Göttsche–Soergel [GS93] is the existence of a canonical isomorphism of $\mathbb{Q}$-Hodge structures:

$$H^*(X) = \bigoplus_{\alpha+n} H^*(S^{(a_1)} \times \cdots \times S^{(a_n)}).$$

(3.15)

Here $S^{(a)} = S^n/\mathfrak{S}_a$ denotes the $a$-th symmetric power of $S$. We omitted the Tate twistings for simplicity, but their isomorphism captures the weight as well. Now using the Hodge structure isomorphism $H^*(S^{(a_i)}) = \text{Sym}^{a_i} W$, we see that the desired identity holds at the level of Hodge structures.

In [GS93], $S$ is assumed to be a polarized K3 surface. The algebraicity on $S$ is not necessary as shown by de Cataldo–Migliorini [dCM00]. This allows us to assume that the Mumford–Tate algebra of $S$ is as big as possible, i.e. $\mathfrak{m}(S) \cong \mathfrak{so}(3, 19)$.

This isomorphism (3.15) gives that the Mumford–Tate algebras $\bar{\mathfrak{m}}(S) = \bar{\mathfrak{m}}(X)$ coincide. Indeed, the Hodge structure of $H^*(X)$ is obtained from a suitable tensor construction applied to the Hodge structure $W = H^*(S)$. By [Moo99, Rem 1.8], the special Mumford-Tate algebra of a tensor construction of $W$ is an image of the special Mumford-Tate algebra of $W$. This means we have a surjection $\bar{\mathfrak{m}}(S) \twoheadrightarrow \bar{\mathfrak{m}}(X)$. On the other hand, choosing $\alpha = (a_1 = 0, \cdots, a_{n-1} = 0, a_n = 1)$, we have a component $L$ on the right hand side. This means $\bar{\mathfrak{m}}(S) \subset \bar{\mathfrak{m}}(X)$. Thus we must have $\bar{\mathfrak{m}}(X) = \bar{\mathfrak{m}}(S)$ by dimension reasons. We write $\bar{\mathfrak{m}}$ for both $\bar{\mathfrak{m}}(S)$ and $\bar{\mathfrak{m}}(X)$, and we understand them as identified via (3.15).
If follows that the identity [GS93] can be interpreted as an $\bar{m}$-module isomorphism. As discussed, we can assume $\bar{m}$ is as large as possible, i.e. $\bar{m} = \bar{g}(S) \cong \mathfrak{so}(3,19)$. Recall that it holds
\[ g(S)_0 = g \oplus \mathbb{R}h \]
where $h$ is the grading operator. Since the isomorphism (3.15) respects the natural grading (when the Tate twists are taken into account), we lift (3.15) to an isomorphism of $g(S)_0$-modules. Since the weight lattices of $g(S)_0$ and $g(S)$ are the same, this is enough to conclude the both hand sides are isomorphic as $g(S)$-modules. (Here the left hand side has a structure of $(X)$-module, which by restriction gives the structure of a $g(S)$-module. While the right hand side only has a natural structure of $g(S)$-module.)

Finally, the $g(X)$-module structure on $H^*(X)$ is in fact uniquely determined by its $g(S)$-module structure. Note that $g(S) = \mathfrak{so}(W,q_S)$ and $g(X) = \mathfrak{so}(V,q)$ are type $D_{12}$ and $B_{12}$ simple Lie algebras. Hence we can apply Proposition B.6.

The above Theorem 3.14 gives us a tool to compute the $g(X)$-module structure of $H^*(X)$, because we can determine its formal character by computing the right hand side of the equality. This method is already very useful to compute the formal character and hence the $g(X)$-module structure of the cohomology of $K3^{[n]}$ type hyper-Kähler manifold. We can make the formula even better by taking care of them all; we consider the generating function of the formal characters of $K3^{[n]}$ hyper-Kähler manifolds. The advantage of this is that we can get rid of the delicate part of partitions in the formula.

Proof of Theorem 3.13. For simplicity, let us write $s_i = \text{ch}(\text{Sym}^i W)$ for the formal character of the symmetric power of the standard $g(S)$-module $W$. By Theorem 3.14, we can write down the generating function by
\[
\sum_{n=0}^{\infty} \text{ch}(K3^{[n]}) q^n = \sum_{n=0}^{\infty} \sum_{\alpha} s_{\alpha} q^n = \sum_{\alpha} s_{\alpha} q^{\alpha_1 + 2 \alpha_2 + \cdots + na_n},
\]
where $\alpha = (\alpha_1, \cdots, \alpha_n)$ runs through all the partitions of $\alpha$. Hence, forgetting about the partition $\alpha$ and just thinking about $\alpha_i$, we can simply rewrite the last expression by
\[
\sum_{\alpha} s_{\alpha} q^{\alpha_1} q^{2 \alpha_2} \cdots q^{na_n} = \left( \sum_{\alpha_1 = 0}^{\infty} s_{\alpha_1} q^{\alpha_1} \right) \left( \sum_{\alpha_2 = 0}^{\infty} s_{\alpha_2} q^{2 \alpha_2} \right) \cdots.
\]
Now setting $A(q) = \sum_{i=0}^{\infty} s_i q^i$, this value is just $A(q) A(q^2) A(q^3) \cdots = \prod_{m=1}^{\infty} A(q^m)$. Moreover, the expression $A(q)$ can be further simplified into
\[
A(q) = \sum_{i=0}^{\infty} s_i q^i = 1 + \text{ch } W q + \text{ch } (\text{Sym}^2 W) q^2 + \cdots = 1 + (x_0 + \cdots + x_1 + x_0^{-1} + \cdots + x_{11}^{-1}) q + (x_0^2 + x_0 x_1 + \cdots + x_{11}^2) q^2 + \cdots
\]
\[
= \prod_{i=0}^{11} (1 + x_i q + x_i^2 q^2 + \cdots)(1 + x_i^{-1} q + x_i^{-2} q^2 + \cdots)
\]
\[
= \prod_{i=0}^{11} \frac{1}{(1 - x_i q)(1 - x_i^{-1} q)}.
\]
The theorem follows.}

3.3. Cohomology of generalized Kummer varieties. In this subsection, we similarly prove the rational coefficient version of the second statement of Theorem 1.1.
**Theorem 3.17.** Let $\mathfrak{g}$ be the Looijenga–Lunts–Verbitsky algebra of hyper-Kähler manifold of $\text{Kum}_n$ type. Let us define the formal power series

$$
B(q) = \prod_{m=1}^{\infty} \left( \prod_{i=0}^{3} \frac{1}{(1 - x_i q^n)(1 - x_i^{-1} q^n)} \prod_{j} \left( 1 + x_0 x_1^j x_2^j x_3 q^m \right) \right),
$$

with $j = (j_0, \ldots, j_3) \in \{-\frac{1}{2}, \frac{1}{2}\} \times \mathbb{Z}$ and $j_0 + \cdots + j_3 \in 2\mathbb{Z}$. Let $b_1$ be the degree 1 coefficient of $B(q) = 1 + b_1 q + b_2 q^2 + \cdots$, and $J_4(d)$ be the fourth Jordan totient function. With these notations, the generating series of the formal characters of the $\mathfrak{g}$-modules $H^*(\text{Kum}_n)$ is

$$
1 + \left( \sum_{i=0}^{3} (x_i + x_i^{-1}) + 16 \right) q + \sum_{n=2}^{\infty} \text{ch}(H^*(\text{Kum}_n)) q^n = \sum_{d=1}^{\infty} J_4(d) \frac{B(q^d) - 1}{b_1 \cdot q}.
$$

**Remark 3.20.** Considering the degree $n$ terms of the identity (3.19), we obtain

$$
\text{ch}(H^*(\text{Kum}_n)) = \frac{1}{b_1} \sum_{d|n+1} J_4 \left( \frac{n+1}{d} \right) \cdot b_d,
$$

where $b_d$ are the coefficients of $B(q) = 1 + b_1 q + b_2 q^2 + \cdots$ given by (3.18). In particular, if $n + 1 = p$ is prime, then

$$
\text{ch}(H^*(\text{Kum}_{p-1})) = \frac{b_p}{b_1} + J_4(p) = \frac{b_p}{b_1} + (p^4 - 1)
$$

has a simple expression. As previously mentioned, the constant term $p^4 - 1$ is an indicator of the trivial representations in $H^*(\text{Kum}_{p-1})$.

For the proof of Theorem 3.17, we follow the same strategy as for $K3^{[n]}$. The only difference here is that the Hodge structure of the generalized Kummer varieties is much more complicated than that of the Hilbert scheme of K3 surfaces, essentially because of the existence of the odd cohomology. Fortunately, the first step, interpreting the Göttsche–Soergel [GS93] result in our language, is fairly straightforward.

**Theorem 3.21.** Let $A$ be a complex torus of dimension 2 and $X$ be the generalized Kummer variety associated to $A$. Write $W = H^*_\text{even}(A)$ and $U = H^*_\text{odd}(A)$ as the standard and half-spin $\mathfrak{g}(A)$-modules. Then the $\mathfrak{g}(X)$-module structure on $H^*(X)$ is uniquely determined by the $\mathfrak{g}(S)$-module isomorphism

$$
H^*(X) \otimes (W \oplus U) = \bigoplus_{\alpha \vdash n+1} \left[ \bigotimes_{i=1}^{n+1} \left( \bigoplus_{j=0}^a \text{Sym}^{a-j} W \otimes \wedge^j U \right) \right]^{\oplus g(\alpha)^4},
$$

where $\alpha = (1^{a_1}, \ldots, (n+1)^{a_{n+1}})$ runs through all the partitions of $n+1 = \sum_{i=1}^{n+1} i a_i$, and $g(\alpha)$ is defined by

$$
g(\alpha) = \text{gcd}\{k : 1 \leq k \leq n+1, \ a_k \neq 0\}.
$$

**Proof.** Again, the following isomorphism is proved in [GS93] on the level of $\mathbb{Q}$-Hodge structures.

$$
H^*(X \times A) = \bigoplus_{\alpha \vdash n+1} H^*(A^{(a_1)} \times \cdots \times A^{(a_{n+1})})^{\oplus g(\alpha)^4}.
$$

Here $A^{(a)} = A^a/\mathfrak{S}_a$ indicates the symmetric power of $A$. Since $A$ in this case has an odd cohomology, the Hodge structure of $A^{(a)}$ has a more complicated form

$$
H^*(A^{(a)}) = \bigoplus_{j=0}^a \text{Sym}^{a-j} W \otimes \wedge^j U.
$$

The proof of it can be found, for example, in [MSS11].

In Theorem 3.14, as K3 surfaces are also hyper-Kähler manifolds, we could avoid the discussion of Looijenga–Lunts–Verbitsky algebra and Mumford–Tate algebra of them. In this case, since $A$ is not a hyper-Kähler manifold, we first need to (1) compute the Looijenga–Lunts–Verbitsky algebra of $A$ and (2) compute the special Mumford–Tate algebra of $A$. Fortunately, both issues can be handled without much
difficulty. First, Looijenga and Lunts [LL97] already computed the Looijenga–Lunts–Verbitsky algebra for an arbitrary complex torus:

\[ g(A) \cong \mathfrak{so}(H^1(A, \mathbb{Q}) \oplus (H^1(A, \mathbb{Q}))') \]

where \((,\)) is the canonical pairing. For dimension 2, this coincides with \(\mathfrak{so}(U^\oplus 4) = \mathfrak{so}(H^2(A, \mathbb{Q}) \oplus \mathbb{Q}^2, q_A \oplus (0 1))\). Hence, the theory of Looijenga–Lunts–Verbitsky algebra of complex tori of dimension 2 coincides with that of hyper-Kähler manifolds (with \(b_2 = 6\)). Second, the special Mumford–Tate Lie algebra of \(H^*(A, \mathbb{Q})\) is that of \(H^1(A, \mathbb{Q})\) because \(H^*(A, \mathbb{Q}) = \wedge^* H^1(A, \mathbb{Q})\). This also coincides with the special Mumford–Tate algebra of \(H^2(X, \mathbb{Q}) = \wedge^2 H^1(A, \mathbb{Q})\), which is a Hodge structure of K3 type, so we can also apply the same argument for complex tori of dimension 2.

Now lifting the Hodge structure isomorphism to an \(m\)-module isomorphism can be done as in the proof of Theorem 3.14. Also, using the Torelli theorem for complex tori, we can vary the complex structure of \(A\) to enhance this isomorphism to a \(g(A)\)-module isomorphism. (Again, we have made use of [dCM00] to be able to work with non-projective complex tori.)

The second part of the theorem requires a new idea. This is because of the additional wedge product terms appearing in Theorem 3.21, and also because of the delicate term \(g(\alpha)^4\).

**Proof of Theorem 3.17.** Write \(s_i = \text{ch}(\text{Sym}^i W)\) and \(w_i = \text{ch}(\wedge^i U)\). Observe that \(U\) is of dimension 8, so we have only \(w_1, \ldots, w_8\). By Theorem 3.21, we can directly compute

\[
\sum_{n=0}^{\infty} \text{ch}(H^*(\text{Kum}_n))(s_1 + w_1)q^{n+1} = \sum_{n=0}^{\infty} \sum_{\alpha \vdash n+1} g(\alpha)^4(s_{a_1} + s_{a_1-1}w_1 + s_{a_1-2}w_2 + \cdots) \cdots (s_{a_{n+1}} + s_{a_{n+1}-1}w_1 + s_{a_{n+1}-2}w_2 + \cdots)q^{n+1} = \sum_{\alpha \neq 0} g(\alpha)^4(s_{a_1} + s_{a_1-1}w_1 + \cdots)q^{a_1}(s_{a_2} + s_{a_2-1}w_1 + \cdots)q^{2a_2}(s_{a_3} + s_{a_3-1}w_1 + \cdots)q^{3a_3} \cdots.
\]

Here in the last expression, \(\alpha = (1^{a_1}, 2^{a_2}, \ldots)\) runs through all nonempty partition and we used \(n+1 = a_1 + 2a_2 + \cdots\). Now, as we did in the proof of Theorem 3.13, we would like to transform the expression without involving the partition \(\alpha\). This cannot be in the same way because of the problematic term \(g(\alpha)^4\). Let us introduce a set \(K = \{k : a_k \neq 0\}\) associated to the partition \(\alpha\). This set is nonempty because \(\alpha\) cannot be an empty partition. Now running \(\alpha\) through all nonempty partition corresponds to running \(K\) for all nonempty finite subset of \(Z\), and varying the multiplicity \(a_k \in Z_{\geq 1}\) for all elements \(k \in K\). Moreover, the notation \(g(\alpha)\) is converted simply to \(\gcd(K)\). Hence, we can convert the last expression by

\[
\sum_{\alpha \neq 0} g(\alpha)^4(s_{a_1} + s_{a_1-1}w_1 + \cdots)q^{a_1}(s_{a_2} + s_{a_2-1}w_1 + \cdots)q^{2a_2}(s_{a_3} + s_{a_3-1}w_1 + \cdots)q^{3a_3} \cdots = \sum_{K} \left(\sum_{(a_k)_{k \in K}} \prod_{k \in K} \gcd(K)^4(s_{a_k} + s_{a_k-1}w_1 + \cdots)q^{k a_k} = \sum_{K} \gcd(K)^4 \prod_{(a_k)_{k \in K}} (s_{a_k} + s_{a_k-1}w_1 + \cdots)q^{k a_k},\right.
\]

where the tuple \((a_k)_{k \in K}\) runs through all the possible functions \(K \to Z_{\geq 1}\). Now one can apply the same factorization technique we used in (3.16) to simplify the last expression into

\[
\sum_{(a_k)_{k \in K}} (s_{a_k} + s_{a_k-1}w_1 + \cdots)q^{k a_k} = \prod_{k \in K} ((s_1 + w_1)q^k + (s_2 + s_1 w_1 + w_2)q^{2k} + \cdots).
\]

For simplicity, let us define \(A(q) = (s_1 + w_1)q + (s_2 + s_1 w_1 + w_2)q^{2} + \cdots\). Then we can write down the last expression in (3.22) simply by

\[
\sum_{K} \gcd(K)^4 \prod_{k \in K} A(q^k).
\]

It is surprising to observe this expression admits a further simplification.
Lemma 3.23. Let $A(q)$ be an arbitrary formal power series on $q$. Then we have an identity

$$
\sum_K \gcd(K)^4 \prod_{k \in K} A(q^k) = \sum_{d=1}^{\infty} J_4(d)(B(q^d) - 1),
$$

where $K$ runs through all the nonempty finite subset of $\mathbb{Z}$, $J_4(d)$ denotes the fourth Jordan totient function and $B(q) = (1 + A(q))(1 + A(q^2))(1 + A(q^3)) \cdots$.

Proof of Lemma 3.23. Let $S^1 = \mathbb{R}/\mathbb{Z}$ be the circle group and $T^4 = (S^1)^4$ the 4-torus. We define the following character function $\delta_k : T^4 \to \{0, 1\}$ of the $k$-th lattice on $T^4$

$$
\delta_k(x, y, z, w) = \begin{cases} 
1 & \text{if } x, y, z, w \in (\frac{1}{k}\mathbb{Z})/\mathbb{Z}, \\
0 & \text{otherwise}.
\end{cases}
$$

Let $d\mu$ be the counting measure on $T^4$. The idea is to capture the nuisance term $\gcd(K)^4$ by the integration of a multiplication of the character functions

$$
\gcd(K)^4 = \int_{T^4} \left( \prod_{k \in K} \delta_k \right) d\mu.
$$

Using this, the left hand side of the identity can be transformed in the following way.

$$
\sum_{K \neq \emptyset} \gcd(K)^4 \prod_{k \in K} A(q^k) = \int_{T^4} \left[ \sum_{K \neq \emptyset} \left( \prod_{k \in K} A(q^k)\delta_k \right) \right] d\mu
$$

$$
= \int_{T^4} \left[ -1 + (1 + A(q)\delta_1)(1 + A(q^2)\delta_2)(1 + A(q^3)\delta_3) \cdots \right] d\mu.
$$

Let us compute this last expression by evaluating the integral at the $d^4$-lattice points $((\frac{1}{k}\mathbb{Z})/\mathbb{Z})^4 \subset T^4$, inductively starting from the lower values of $d$. At the point $(\frac{x_1}{d}, \cdots, \frac{x_4}{d}) \in T^4$, if $\gcd(c_1, c_2, c_3, c_4, d) > 1$, then this point was already counted when we considered the $(d')^4$-lattice points with $d' < d$. When $\gcd(c_1, c_2, c_3, c_4, d) = 1$, the evaluation of the integral at this point gives precisely the formal power series

$$
-1 + (1 + A(q^d))(1 + A(q^{2d}))(1 + A(q^{3d})) \cdots = B(q^d) - 1.
$$

The number of points $(\frac{x_1}{d}, \cdots, \frac{x_4}{d})$ with $\gcd(c_1, c_2, c_3, c_4, d) = 1$ is by definition the fourth Jordan totient function value $J_4(d)$. This proves the lemma.

Returning to the proof of Theorem 3.17, we now have

$$
\sum_{n=0}^{\infty} \text{ch}(H^*(\text{Kum}_n))(s_1 + w_1)q^{n+1} = \sum_{d=1}^{\infty} J_4(d)(B(q^d) - 1).
$$

In our case, we have the further identities

$$
1 + A(q) = 1 + (s_1 + w_1)q + (s_2 + s_1w_1 + w_2)q^2 + (s_3 + s_2w_1 + s_1w_2 + w_3)q^3 + \cdots
$$

$$
= (1 + s_1q + s_2q^2 + \cdots)(1 + w_1q + \cdots + w_8q^8)
$$

$$
= \frac{1}{\prod_{i=0}^{3}(1 - x_iq)(q - x_i^{-1}q)} \prod_{j}(1 + x_{j0}x_{j1}x_{j2}x_{j3}q).
$$

We finally note $A(q) = 1 + (s_1 + w_1)q + \cdots$ so that $B(q) = (1 + A(q))(1 + A(q^2)) \cdots = 1 + (s_1 + w_1)q + \cdots$. Hence we can set $s_1 + w_1 = b_1$, where $b_1$ denotes the first $q$-coefficient of $B(q)$. This completes the proof of the theorem. \qed
3.4. Cohomology of O’Grady’s 10-dimensional example. The case of hyper-Kähler manifolds of OG10 type is in some sense the most interesting case, as it shows the power of the LLV decomposition of the cohomology. To start, we recall the very recent result of de Cataldo–Rapagnetta–Saccà [dCRS19].

**Theorem 3.24** (de Cataldo–Rapagnetta–Saccà [dCRS19]). Let $X$ be a hyper-Kähler manifold of OG10 type. Then

i) There is no odd cohomology ($H^*_{\text{odd}}(X) = 0$).

ii) The Hodge numbers of $H^*_{\text{even}}(X)$ are as follows (we list only the first quadrant):

\[
\begin{array}{cccc}
1 & 22 & 1 & 1 \\
254 & 22 & 1 & 1 \\
2,299 & 276 & 23 & 1 \\
16,490 & 2,531 & 276 & 22 & 1 \\
88,024 & 16,490 & 2,299 & 254 & 22 & 1 \\
\end{array}
\] (3.25)

While in the other cases we have made heavy use of the knowledge of the Hodge numbers, it turns out that in the OG10 case the existence of the LLV decomposition with respect to $\mathfrak{so}(4,22)$ is a very constraining condition. In particular, we obtain that item (ii) of Theorem 3.24 is a corollary of item (i). To emphasize this fact, we state the following somewhat artificial result:

**Theorem 3.26.** Let $X$ be a 10-dimensional hyper-Kähler manifold. Assume the following

1) $b_2(X) = 24$;

2) $e(X) = 176,904$.

3) There is no odd cohomology ($H^*_{\text{odd}}(X) = 0$).

Then $X$ has the following LLV decomposition as a $\mathfrak{g} = \mathfrak{so}(4,22)$-module:

$H^*(X) = V_5 \oplus V_{(2,2)}$. (3.27)

In particular, the Hodge numbers are as in (3.25).

**Remark 3.28.** An alternative notation for this result, the one written in the introduction, is $H^*(X) = V_{5\varpi_1} \oplus V_{2\varpi_2}$. Here $\varpi_1$ is the fundamental weight associated to the standard representation $V$ (and thus $V_{5\varpi_1}$ is the leading representation in $\text{Sym}^5 V$), and $\varpi_2$ is the fundamental weight associated to the irreducible $\mathfrak{g}$-module $\wedge^2 V$.

**Corollary 3.29.** If $X$ is a hyper-Kähler manifold of OG10 type, then its LLV decomposition is given by (3.27) and the Hodge numbers are as in (3.25).

**Proof.** The three conditions of Theorem 3.26 were established by Rapagnetta [Rap08], Mozgovoy [Moz06] (see also [HLS19]), and de Cataldo–Rapagnetta–Saccà [dCRS19] (Theorem 3.24(i)) respectively. □

**Remark 3.30.** It is interesting to note that in the OG10 case, the vanishing of the odd cohomology is equivalent to Theorem 1.14. Specifically, assuming no odd cohomology, we obtain the LLV decomposition (3.27), which obviously satisfies the condition (1.15) in Theorem 1.14. Conversely, assuming (1.15), we conclude that there is no odd cohomology. Namely, for OG10, the rank of the LLV algebra $\mathfrak{g}$ is 13. Any irreducible $\mathfrak{g}$-module occurring in the odd cohomology $V_\mu \subset H^*_{\text{odd}}(X,\mathbb{Q})$ has all the coefficients of $\mu = (\mu_0,\cdots,\mu_{12})$ half-integers. But then,

$$\mu_0 + \cdots + \mu_{11} + |\mu_{12}| \geq \frac{13}{2} > 5,$$

violating the the inequality (1.15). The same argument applies more generally. Namely, the condition (1.15) forces the vanishing of odd cohomology for $2n$-dimensional hyper-Kähler manifolds satisfying

$$\left\lfloor \frac{b_2}{2} \right\rfloor > 2n - 1.$$ (3.31)
The rest of the section is concerned with the proof of Theorem 3.26. In addition to the numerical assumptions of the theorem, we are using the following three general facts about the cohomology of hyper-Kähler manifolds.

(A) $H^*(X)$ admits an action by the LLV algebra $\mathfrak{g} \cong \mathfrak{so}(4, b_2 - 2)$. In this situation, the assumptions of Theorem 3.26 give $\mathfrak{g} \cong \mathfrak{so}(4, 22)$, $H^*(X) = H_{\text{even}}(X)$ has dimension 176,904. The main point here is that this dimension is relatively small with respect to $\mathfrak{g}$.

(B) The Verbitsky component $V_{(5)}$ occurs in $H^*(X)$. Since $\dim V_{(5)} = 139,230$, we obtain that the other irreducible representations occurring in $H^*(X)$ have total dimension 37,674.

(C) A $2n$-dimensional hyper-Kähler manifold satisfies Salamon’s relation:

$$2 \sum_{i=1}^{2n} (-1)^i (3i^2 - n)b_{2n-i} = nb_{2n},$$

which we find more convenient to rewrite it as $\sum_{i=1}^{2n} (-1)^i i^2 b_{2n-i} = \frac{n}{6} \cdot e(X)$. Assuming no odd cohomology, we get

$$\sum_{k=0}^{n} (n-k)^2 b_{2k} = \frac{n}{24} \cdot e(X) \quad (3.32)$$

(e.g. in dimension $2 = 2n$, this reads $b_0 = \frac{e(X)}{24}$, which is equivalent to Noether’s formula for hyper-Kähler $[\equiv K3]$ surfaces).

In the particular case considered here, we obtain the following four equations for the six even Betti numbers:

- $b_0 = 1$
- $b_2 = 24$
- $25b_0 + 16b_2 + 9b_4 + 4b_6 + b_8 = 36,855$
- $2b_0 + 2b_2 + 2b_4 + 2b_6 + b_8 + b_{10} = 176,904$

There are finitely many non-negative integer solutions $b_{2i}$ to the above equations. It turns out that there is a unique solution compatible with the LLV structure. Specifically, after base-changing to $\mathbb{C}$, we have

$$H^*_{\text{even}}(X) = V_{(5)} \oplus V' \quad (3.33)$$

for some $\mathfrak{so}(4,22)$-module $V'$. The dimension bound discussed above, greatly limits the possibilities for the irreducible summands of $V'$.

**Lemma 3.34.** The possible dominant integral $\mathfrak{so}(4, 22)$-weights $\mu$ such that $V_\mu$ can be contained in $V'$ are $S = \{(4), (3), (2, 2), (2, 1), (2), (1, 1, 1, 1), (1, 1, 1), (1, 1), (1), (0)\}$.

**Proof.** As discussed, $\dim V' = 37,674$. On the other hand, by Proposition 2.35, $\mu = (\mu_0, \cdots, \mu_{12})$ has integer coefficients $\mu_i$. Using Weyl dimension formula and Lemma A.9, one can check that $S$ is the complete list of for the dominant weights for the $\mathfrak{so}(4, 22)$-modules satisfying these constraints. \qed

As discussed in Section 2, each of the $\mathfrak{so}(4, 22)$-modules $V_\mu$ carry a Hodge structure (induced by $h, f \in \mathfrak{g} = \mathfrak{so}(4, 22)$), and hence each $V_\mu$ admits its own Betti numbers. We list the relevant Betti numbers in Table 1. Writing

$$V' = \bigoplus_{\mu \in S} V_{\mu}^{\oplus m_\mu},$$

for the irreducible decomposition of $V'$, and using Table 1, we obtain the following constraints:

i) $m_{(4)} = 0$ (since $b_2 = 24$).

ii) (Euler characteristic)

$$3,250m_{(3)} + 37,674m_{(2,2)} + 5,824m_{(2,1)} + 350m_{(2)} + 14,950m_{(1,1,1,1)} + 2,600m_{(1,1,1)} + 325m_{(1,1)} + 26m_{(1)} + m_0 = 37,674 \quad (3.35)$$

iii) (Salamon’s relation)

$$405m_{(3)} + 5,796m_{(2,2)} + 672m_{(2,1)} + 28m_{(2)} + 2,024m_{(1,1,1,1)} + 276m_{(1,1,1)} + 24m_{(1,1)} + m_{(1)} = 5,796 \quad (3.36)$$

It turns out that this system of equations has a unique (obvious) solution.
Lemma 3.37. The above equations admit a unique nonnegative integer solution
\[ m_{(2,2)} = 1, \quad m_{(4)} = m_{(3)} = m_{(2,1)} = \cdots = m_{(0)} = 0. \]

Proof. By dimension reasons, \( m_{(2,2)} \geq 1 \) forces the solution listed in the lemma. Thus, we can assume \( m_{(2,2)} = 0 \). By rescaling the Euler characteristic equation (3.35) by \( \frac{1}{13} \), we get
\[
500m_{(3)} + 896m_{(2,1)} + \left(53 + \frac{11}{13}\right)m_{(2)} + 2,300m_{(1,1,1,1)} + 400m_{(1,1,1)} + 50m_{(1,1)} + 2m_{(1)} + \frac{2}{13}m_{(0)} = 5,796.
\]
We notice that the coefficients of this equation are all larger than the corresponding ones in the Salamon’s relation (3.36) (while the value on the right-hand-side stays the same). We conclude that there is no non-negative solution (with \( m_{(2,2)} = 0 \)). The lemma follows. \( \square \)

This concludes the proof of Theorem 3.26 (and thus Theorem 1.1(4)).

3.5. Cohomology of O’Grady’s 6-dimensional example. We now prove the OG6 case of Theorem 1.1. Specifically, we prove:

Theorem 3.38. Let \( X \) be a hyper-Kähler manifold of OG6 type and \( g \cong \mathfrak{so}(4,6) \) its Looijenga–Lunts–Verbitsky algebra. Then the \( g \)-module irreducible decomposition of the cohomology of \( X \) is
\[
H^*(X) = V_{(3)} \oplus V_{(1,1,1)} \oplus V_{(1)} \oplus 240.
\]

Remark 3.39. An alternative notation for this result, the one written in the introduction, is \( H^*(X) = V_{\varpi_1} \oplus V_{\varpi_3} \oplus V^{\oplus 135} \oplus \mathbb{R}^{\oplus 240} \). Here \( \varpi_1 \) is the fundamental weight associated to the standard representation \( V \) and \( \varpi_3 \) is the fundamental weight associated to the irreducible \( g \)-module \( \wedge^3 V \).

The starting point of our result are the Hodge numbers computed by Mongardi–Rapagnetta–Saccà [MRS18]. Again, there is no odd cohomology, and the relevant Hodge numbers are given in Table 2.

<table>
<thead>
<tr>
<th>b_0</th>
<th>b_2</th>
<th>b_4</th>
<th>b_6</th>
<th>b_8</th>
<th>b_{10}</th>
<th>Dimension</th>
<th>( \sum (5 - i)^2 b_{2i} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>V_{(5)}</td>
<td>1</td>
<td>24</td>
<td>300</td>
<td>2,600</td>
<td>17,550</td>
<td>98,280</td>
<td>139,230</td>
</tr>
<tr>
<td>V_{(4)}</td>
<td>1</td>
<td>24</td>
<td>300</td>
<td>2,600</td>
<td>17,550</td>
<td>23,400</td>
<td>4,032</td>
</tr>
<tr>
<td>V_{(3)}</td>
<td>1</td>
<td>24</td>
<td>300</td>
<td>2,600</td>
<td>3,250</td>
<td>405</td>
<td></td>
</tr>
<tr>
<td>V_{(2,2)}</td>
<td>299</td>
<td>4,600</td>
<td>27,876</td>
<td>37,674</td>
<td>5,796</td>
<td></td>
<td></td>
</tr>
<tr>
<td>V_{(2,1)}</td>
<td>24</td>
<td>576</td>
<td>4,624</td>
<td>5,824</td>
<td>672</td>
<td></td>
<td></td>
</tr>
<tr>
<td>V_{(2)}</td>
<td>1</td>
<td>24</td>
<td>300</td>
<td>350</td>
<td>28</td>
<td></td>
<td></td>
</tr>
<tr>
<td>V_{(1,1,1,1)}</td>
<td>2,024</td>
<td>10,902</td>
<td>14,950</td>
<td>2,024</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>V_{(1,1,1)}</td>
<td>276</td>
<td>2,048</td>
<td>2,600</td>
<td>276</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>V_{(1,1)}</td>
<td>24</td>
<td>277</td>
<td>325</td>
<td>24</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>V_{(1)}</td>
<td>1</td>
<td>24</td>
<td>26</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>V_{(0)}</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 1. The relevant irreducible \( \mathfrak{so}(4,22) \)-modules for OG10

Table 2. The Hodge diamond of OG6, the component \( V_{(3)} \), and the residual component \( V' \).

Splitting off the Verbitsky component from the cohomology of OG6 hyper-Kähler manifold \( X \)
\[
H^*(X) = V_{(3)} \oplus V',
\]
it remains to understand the residual component $V'$ (see Table 2 for the numerics). We proceed as for the OG10 case. Unfortunately, it turns out that there are two possible solutions to the numerical constraints satisfied by OG6 (even when the Hodge numbers are taken into account).

**Proposition 3.40.** The LLV decomposition of $H^*(\text{OG6})$ as a $\mathfrak{so}(4,6)$-module is either

$$H^*(X) = V_{(3)} \oplus V_{(1,1)} \oplus V^{\oplus 135} \oplus \mathbb{R}^{\oplus 240} \quad \text{or} \quad V_{(3)} \oplus V^{\oplus 6} \oplus V^{\oplus 115} \oplus \mathbb{R}^{\oplus 290}. \quad (3.41)$$

**Proof.** Straight-forward manipulations of the Hodge numbers, similar to the OG10 case (see §3.4). We omit the details. □

In order to decide which of the two possibilities of (3.41) actually occurs in the LLV decomposition of the OG6 example, we need to investigate further the geometric construction of [MRS18]. First, it is not hard to lift the computations of Hodge numbers in loc. cit. to a statement about Hodge structures (Proposition 3.44). This allows us to understand the decomposition with respect to the Mumford–Tate algebra $\bar{\mathfrak{m}}$ of $H^4(X)$ (Proposition 3.46). Finally, we complete the proof in §3.5.3, by considering the possible restriction representations of the two cases in (3.41) from $\mathfrak{g}$-representations to $\bar{\mathfrak{m}}$-representations (recall $\mathfrak{m} \subset \bar{\mathfrak{g}} \subset \mathfrak{g}$).

**Remark 3.42.** There are two heuristic reasons why the situation is more complicated in the OG6 case versus the OG10 case. First the LLV algebra is much smaller in this case $\mathfrak{so}(4,6)$ (vs. $\mathfrak{so}(4,22)$). Secondly, OG6 is an exceptional case of the Kum$_n$ series, meaning that multiple trivial representations will occur, which in turn means less rigidity for the numerical constraints.

3.5.1. **Review of [MRS18] construction.** Let $X$ be a hyper-Kähler manifold of OG6 type. The basic topological invariants of $X$ were found by Rapagnetta [Rap07] by realizing $X$ as the resolution of the quotient of some companion $K3^{[3]}$ hyper-Kähler manifold $Y$ by a birational involution $\iota$. This model was then used by Mongardi–Rapagnetta–Sacca [MRS18] for the computation of Hodge numbers. We review their construction, and extract some further consequences.

Let $X = Y/\iota$ as above (N.B. since the involution is only birational, the equality should be understood as contacting $Y$ to a singular model on which the involution is regular, followed by a symplectic resolution of the quotient). To avoid working with birational involutions and singular models, one considers a blow-up $\hat{Y}$ of $Y$ on which the involution lifts to a regular involution $\iota$. The quotient $\hat{X} = \hat{Y}/\iota$ is a blow-up of the OG6 manifold $X$. More specifically, one has the following diagram:

$$
\begin{array}{ccc}
\hat{Y} & \xrightarrow{\iota} & \hat{X} \\
\downarrow \text{blowup } \Delta & & \downarrow \text{blowdown 256 quadrics} \\
\hat{Y} & \xrightarrow{\iota} & X
\end{array}
$$

The following facts (cf. [Rap07, MRS18]) will be needed in our arguments:

(0) Let $A$ be a very general principally polarized abelian surface. Let $S = S(A) = \widetilde{A/\pm 1}$ be the Kummer K3 surface associated to $A$ (it contains 16 disjoint $\mathbb{P}^1$). The $K3^{[3]}$ hyper-Kähler manifold $Y$ is birational to $S^{[3]}$, and it contains 256 disjoint $\mathbb{P}^3$. The OG6 manifold $X$ is obtained as a moduli sheaves on $A$ and resolving (as in [O’G03]). By construction there is a birational involution $\iota$ on $Y$ such that birationally $Y/\iota \cong_{\text{bir}} X$.

(1) The blow-up \( \hat{Y} \to Y \) is the composition of the blow-up $\tilde{Y}$ of the 256 copies of $\mathbb{P}^3$ in $Y$, followed by the blow-up of the strict transform $\Delta \subset \tilde{Y}$ of a certain diagonal locus.

(2) The center $\Delta$ of the blow-up $\hat{Y} \to \tilde{Y}$ is smooth and isomorphic to the blow-up of 256 nodes of $(A \times A^\vee)/\pm 1$. In particular, the exceptional divisor $\Delta \subset \tilde{Y}$ is a $\mathbb{P}^1$ bundle over $\Delta$.

(3) The involution $\iota$ lifts to a regular involution $\iota$ on $\hat{Y}$. The quotient variety $\hat{X} = \hat{Y}/\iota$ is smooth, and the divisor $\Delta \subset \hat{Y}$ is $\iota$-invariant.

(4) The OG6 manifold $X$ is obtained from $\hat{X}$ by contracting 256 disjoint smooth threefolds, each isomorphic to a quadric threefold.
In [MRS18], the Hodge numbers of $X$ are obtained from the knowledge of the Hodge numbers of $Y$ and tracing through the diagram (3.43) using the above mentioned facts. While not explicitly mentioned in loc. cit., a careful reading of [MRS18, Sect. 6] gives the following statement about the relationship between the $\mathbb{Q}$-Hodge structures and Mumford–Tate algebras, it is necessary to work with the field $\mathbb{Q}$.

**Proposition 3.44.** There exists an isomorphism of $\mathbb{Q}$-Hodge structures

$$H^*(X, \mathbb{Q}) = H^*(Y, \mathbb{Q})^\sigma \oplus H_{\text{even}}^*(A \times A^\vee, \mathbb{Q})(-1) \oplus 256\mathbb{Q}(-3).$$

Here $H^*(Y, \mathbb{Q})^\sigma$ indicates the invariant cohomology of $H^*(Y, \mathbb{Q})$ with respect to an appropriate involution $\sigma$.

**Proof.** This follows from Section 6 of [MRS18]. Their statements are formulated in terms of Hodge numbers, but in fact all their proofs apply at the level of Hodge structures. First, by [MRS18, Lemma 6.2(1)] we get

$$H^*(\tilde{X}, \mathbb{Q}) = H^*(X, \mathbb{Q}) \oplus 256\mathbb{Q}(-1) \oplus 512\mathbb{Q}(-2) \oplus 512\mathbb{Q}(-3) \oplus 512\mathbb{Q}(-4) \oplus 256\mathbb{Q}(-5).$$

Lemmas 6.2(2) and 6.3 in [MRS18] give the Hodge structure isomorphism

$$H^*(\tilde{X}, \mathbb{Q}) = H^*(\tilde{Y}, \mathbb{Q})^\tau \oplus H_{\text{even}}^*(A \times A^\vee, \mathbb{Q})(-1) \oplus 256\mathbb{Q}(-2) \oplus 256\mathbb{Q}(-3) \oplus 256\mathbb{Q}(-4).$$

Finally, [MRS18, Lemma 6.5(2)] states the Hodge structure isomorphism

$$H^*(\tilde{Y}, \mathbb{Q})^\tau = H^*(Y, \mathbb{Q})^\tau \oplus 256\mathbb{Q}(-1) \oplus 256\mathbb{Q}(-2) \oplus 512\mathbb{Q}(-3) \oplus 256\mathbb{Q}(-4) \oplus 256\mathbb{Q}(-5).$$

Combining the three isomorphisms, we get the desired identification.

3.5.2. The Mumford–Tate decomposition. Recall that $A$ was a very general principally polarized abelian surface. Hence, the Mumford–Tate algebra of $A$ is

$$\m = \mathfrak{so}(\tilde{T}, \tilde{q}_A),$$

where $\tilde{T} \subset H^2(A, \mathbb{Q})$ is the transcendental Hodge structure of the second cohomology of $A$ and $\tilde{q}_A$ is the intersection form. Note that $\dim_{\mathbb{Q}} \tilde{T} = 5$ and the signature of $\tilde{q}_A$ is $(2, 3)$. This implies the real form of $\m$ is isomorphic to $\mathfrak{so}(2, 3)$.

**Lemma 3.45.** The special Mumford–Tate algebras of $X$ and $Y$ are both isomorphic to $\m$.

**Proof.** Since $Y$ is obtained from the Kummer surface $S = A/\pm 1$, the statement is standard. Using Proposition 3.44, the statement follows also for $X$. We omit further details.

Using Lemma 3.45 and a more careful inspection of the involution $\sigma$ from [MRS18, Sect. 6], we obtain the decomposition of the cohomology of $H^4(X, \mathbb{Q})$ as a $\m$-module.

**Proposition 3.46.** Let $\tilde{W}$ be the standard $\m$-module. Then the fourth cohomology of $X$ has the $\m$-module decomposition

$$H^4(X, \mathbb{Q}) = \tilde{W}(2) \oplus \tilde{W}(1,1) \oplus 6\tilde{W} \oplus 145\mathbb{Q}.$$

**Proof.** Proposition 3.44 gives

$$H^4(X, \mathbb{Q}) = H^4(Y, \mathbb{Q})^\sigma \oplus H^2(A \times A^\vee, \mathbb{Q})(-1).$$

Let us first compute the second component. Applying Künneth and standard representation theory, we obtain

$$H^2(A \times A^\vee, \mathbb{Q}) = H^2(A, \mathbb{Q}) \oplus H^2(A^\vee, \mathbb{Q}) \oplus [H^1(A, \mathbb{Q}) \otimes H^1(A, \mathbb{Q})]$$

$$= 2\tilde{W} \oplus 2\mathbb{Q} \oplus (\tilde{W}_{1,2}^\pm, 4)^{\otimes 2} = \tilde{W}(1,1) \oplus 3\tilde{W} \oplus 3\mathbb{Q}. \quad (3.47)$$

Next, we need to compute $H^4(Y, \mathbb{Q})^\sigma$. To do so, we imitate the trick used in the proof of [MRS18, Lem 6.6]. We first compare the second cohomology of the identification of Proposition 3.44. This gives us the Hodge structure isomorphism

$$H^2(X, \mathbb{Q}) = H^2(Y, \mathbb{Q})^\sigma \oplus \mathbb{Q}(-1).$$
But we already know what the Hodge structure of $H^2(X, \mathbb{Q})$ is by Lemma 3.45 with our old Proposition 2.38(2). Both $H^2(X, \mathbb{Q})$ and $\bar{W}$ are $\bar{m}$-modules and their Hodge numbers are $(1, 6, 1)$ and $(1, 3, 1)$, respectively. This forces an $\bar{m}$-module isomorphism $H^2(X, \mathbb{Q}) = \bar{W} \oplus 3\mathbb{Q}$. Hence, we get $H^2(Y, \mathbb{Q})^\sigma = \bar{W} \oplus 2\mathbb{Q}$ as $\bar{m}$-modules. By similar Hodge number argument, we have $H^2(Y, \mathbb{Q}) = \bar{W} \oplus 18\mathbb{Q}$. Writing $H^2_+(Y, \mathbb{Q}) = H^2(Y, \mathbb{Q})^\sigma$ and $H^2_-(Y, \mathbb{Q})$ by the ±1 eigenspaces of the involution $\sigma$ on $H^2(Y, \mathbb{Q})$, we have $\bar{m}$-module isomorphisms $H^2_+(Y, \mathbb{Q}) = H^2(Y, \mathbb{Q})^\sigma = \bar{W} \oplus 2\mathbb{Q}$, $H^2_-(Y, \mathbb{Q}) = 16\mathbb{Q}$.

The involution $\sigma$ was constructed as a monodromy operator on the space $Y$ (see [MRS18, §6]). Since the monodromy action respects the (reduced) Looijenga–Lunts–Verbitsky algebra $\bar{g}$-structure on each cohomology (cf. [Mar02]), it follows that $H^4(Y, \mathbb{Q}) = \text{Sym}^2 H^2(Y, \mathbb{Q}) \oplus H^2(Y, \mathbb{Q})$ as $\bar{g}(Y)$-modules by the computation in Remark 3.3. This means $H^4(Y, \mathbb{Q})^\sigma$ is precisely $H^4(Y, \mathbb{Q})^\sigma = \text{Sym}^2 H^2_+(Y, \mathbb{Q}) \oplus \text{Sym}^2 H^2_-(Y, \mathbb{Q}) \oplus H^2_+(Y, \mathbb{Q}) = \text{Sym}^2(\bar{W} \oplus 2\mathbb{Q}) \oplus \text{Sym}^2(16\mathbb{Q}) \oplus \bar{W} \oplus 2\mathbb{Q} = \bar{W}(2) \oplus 3\bar{W} \oplus 142\mathbb{Q}$. (3.48)

Combining (3.47) and (3.48), we deduce the result. □

3.5.3. Completion of the proof of Theorem 3.5. We complete the computations of the LLV decomposition in the OG6 case by studying the possible restrictions of the $g$-representations occurring in (3.41) to $\bar{m}$-representations. For reader’s convenience let’s recall the inclusions of algebras $\bar{m} \subset \bar{g} \subset g$, with $g \cong \text{so}(4, 6)$ the LLV algebra, $\bar{g} \cong \text{so}(3, 5)$ the restricted LLV algebra, and finally $m \cong \text{so}(2, 3)$ the Mumford-Tate algebra.

$\bar{m} \subset \bar{g} \subset g$

$\text{so}(2, 3) \subset \text{so}(3, 5) \subset \text{so}(4, 6)$

We also recall that $\bar{g}$ (and thus also $\bar{m}$) respect the cohomological degree. We focus on degree 4 cohomology $H^4(X)$ as the first non-obvious piece for the $\bar{g}$-action. First we investigate the restriction of the two cases of (3.41) from $g$ to $\bar{g}$-modules.

Lemma 3.49. Let $X$ be a hyper-Kähler 6-fold with $b_2(X) = 8$.

(i) Assume that the LLV decomposition of $H^*(X)$ is

$H^*(X) = V_{(3)} \oplus V_{(1,1,1)} \oplus 135V \oplus 240\mathbb{R}$

as $g \cong \text{so}(4, 6)$-modules. Then $H^4(X) = \bar{V}_{(2)} \oplus \bar{V}_{(1,1)} \oplus 136\mathbb{R}$. (3.50)

as $\bar{g} \cong \text{so}(3, 5)$-modules.

(ii) Assume that the LLV decomposition of $H^*(X)$ is

$H^*(X) = V_{(3)} \oplus 6V_{(1,1,1)} \oplus 115V \oplus 290\mathbb{R}$

as $g \cong \text{so}(4, 6)$-modules. Then $H^4(X) = \bar{V}_{(2)} \oplus 6\bar{V} \oplus 116\mathbb{R}$. (3.51)

as $\bar{g} \cong \text{so}(3, 5)$-modules.

Proof. We proceed as in Remark 3.3 (see also Appendix B). Recall that the standard representation $V$ of $g$ is the Mukai completion of the standard representation $\bar{V}$ of $\bar{g}$. Regarding $V$ as a $\bar{g}$ module gives:

$V = \mathbb{R}(1) \oplus \bar{V} \oplus \mathbb{R}(-1)$,
where we indicate the twist to keep track of the cohomological degree. It is immediate to see

\[ V(3) = \mathbb{R}(3) \oplus \tilde{V}(2) \oplus \text{Sym}^2 \tilde{V}(1) \oplus \text{Sym}^3 \tilde{V} \oplus \text{Sym}^2 \tilde{V}(-1) \oplus \tilde{V}(-2) \oplus \mathbb{R}(-3), \]

\[ V_{(1,1)} = \wedge^2 \tilde{V}(1) \oplus [\wedge^3 \tilde{V} \oplus \tilde{V}] \oplus \wedge^2 \tilde{V}(-1), \]

\[ V_{(1,1)} = \tilde{V}(1) \oplus [\wedge^3 \tilde{V} \oplus \mathbb{R}] \oplus \tilde{V}(-1). \]

It follows that

\[ H^*(X) = \mathbb{R}(3) \oplus \tilde{V}(2) \oplus \left[ \text{Sym}^2 \tilde{V} \oplus \wedge^2 \tilde{V} \oplus 135 \mathbb{R} \right] \oplus \left[ \text{Sym}^3 \tilde{V} \oplus \wedge^3 \tilde{V} \oplus 136 \tilde{V} \oplus 240 \mathbb{R} \right] \oplus \cdots, \]

for the first case, and

\[ H^*(X) = \mathbb{R}(3) \oplus \tilde{V}(2) \oplus \left[ \text{Sym}^2 \tilde{V} \oplus 6 \tilde{V} \oplus 115 \mathbb{R} \right] \oplus \left[ \text{Sym}^3 \tilde{V} \oplus 6 \wedge^2 \tilde{V} \oplus 116 \tilde{V} \oplus 296 \mathbb{R} \right] \oplus \cdots, \]

for the second case. The lemma follows. \( \Box \)

Finally, we restrict from \( \mathfrak{g} \)-representation on \( H^4(X) \) to a \( \mathfrak{m} \)-representation.

**Proposition 3.52.** With notations and assumptions as in Lemma 3.49

(i) If (3.50) holds, then

\[ H^4(X) = \tilde{W}(2) \oplus \tilde{W}_{(1,1)} \oplus 6 \tilde{W} \oplus 145 \mathbb{R}, \]

as \( \mathfrak{m} \cong \text{so}(2,3) \)-modules.

(ii) If (3.51) holds, then

\[ H^4(X) = \tilde{W}(2) \oplus 9 \tilde{W} \oplus 140 \mathbb{R}, \]

as \( \mathfrak{m} \cong \text{so}(2,3) \)-modules.

**Proof.** This follows directly from the result of Lemma 3.49 with the decomposition \( \tilde{V} = \tilde{W} \oplus 3 \mathbb{R} \). The latter fact follows from the comparison of the Hodge diamond of \( \tilde{V} \) and \( \tilde{W} \), which are \( (1,6,1) \) and \( (1,3,1) \) respectively. \( \Box \)

**Proof Theorem 3.38.** Using the numerical restrictions on \( \text{OG6} \) type, we have determined two compatible LLV \( \mathfrak{g} = \text{so}(4,6) \)-decompositions of the cohomology (Prop. 3.40). In Proposition 3.52, we have determined the restrictions of these two cases as representations of the Mumford–Tate algebra \( \mathfrak{m} = \text{so}(2,3) \). Only one of them matches the geometric possibility identified in Proposition 3.46. The claim follows. \( \Box \)

### 4. Period maps, monodromy, and the LLV algebra

As discussed in §2.2.1, the Hodge structure on \( H^*(X) \) is determined by two operators \( h, f \in \mathfrak{g} \) (and in particular it factors through the LLV algebra \( \mathfrak{g} \)). We further recall that \( h \in \mathfrak{g}_0 \) and \( f \in \mathfrak{g}_C \), giving the usual Hodge structures on the degree \( k \) cohomology \( H^k(X) \) (via the representations \( \rho_k \) of (2.5)). The purpose of this section is to discuss the behavior of the Hodge structure in families, and more precisely to discuss the higher degree period maps. Given that the Torelli theorem holds, and that the LLV algebra is determined by \( H^2(X) \), it is no surprise that the higher period maps and higher degree operators are determined by those for \( H^2(X) \) (Theorems 4.1 and 4.12). The recent work of Soldatenkov [Sol18, Sol19] covers similar ground; indeed, Theorem 4.12 is [Sol18, Prop. 3.4]. Conceptually, the discussion here is related to the concept of Hodge representations discussed in [GGK12, IV], though the two specific results that we need are not discussed there.

#### 4.1. Higher degree period maps

Given a family of hyper-Kähler manifolds \( \mathcal{X}/S \), for each degree \( k \), one gets periods maps

\[ \Phi_k : \tilde{S} \to D_k \]

from the universal cover \( \tilde{S} \) of \( S \) to a classifying space \( D_k \) of Hodge structures (with specified Hodge numbers, matching those of \( H^k(X_s) \)). Verbitsky’s global Torelli theorem [Ver13, Huy12] says that a hyper-Kähler manifold \( X \) is essentially recovered from its Hodge structure on \( H^2(X) \). Thus, one expects that the \( k \)-th period map \( \Phi_k \) is recovered from the second period map \( \Phi_2 \). We would like to make this precise. To avoid various issues, we will state only an infinitesimal version. To start, we define the period domain to the (compact) symmetric space parameterizing Hodge flags with specified Hodge numbers:

\[ \hat{D}_k = \text{Flag}(H^k(X, C), (f^*)) \],
where \((f^*)\) indicates the dimensions of the Hodge filtration of the \(k\)-th cohomology of the fiber \(X = X_s\).

The general linear group \(\text{GL}(H^k(X, \mathbb{C}))\) acts on \(D_k\) transitively. Fixing a reference point \(o_k \in \tilde{D}_k\) gives the stabilizer \(P_k = \text{Stab}_{\text{GL}(H^k(X, \mathbb{C}))}(o_k)\), and hence an identification
\[
\tilde{D}_k = \text{GL}(H^k(X, \mathbb{C}))/P_k.
\]

The \(k = 2\) case is special, as we take into account the polarization, in this case the Beauville–Bogomolov form on \(H^2(X)\). This means that the structure group becomes an orthogonal group, and we have
\[
\tilde{D}_2 = \text{SO}(\bar{V}, \bar{q})/P_2.
\]

We these preliminaries, we can state our result as follows.

**Theorem 4.1.** There exists a (unique) morphism between the period domains \(\psi_k : \tilde{D}_2 \to \tilde{D}_k\) with the following property. Let \(\mathcal{X}/S\) be a smooth proper family of compact hyper-Kähler manifolds. Let \(\tilde{S}\) be the universal covering of \(S\), and \(\tilde{\Phi}_2 : \tilde{S} \to \tilde{D}_2\) and \(\tilde{\Phi}_k : \tilde{S} \to \tilde{D}_k\) the second and \(k\)-th period maps associated to the family. Then we have \(d\tilde{\Phi}_k = d\psi_k \circ d\tilde{\Phi}_2\) at all points in \(\tilde{S}\).

\[
\begin{array}{ccc}
T_{\tilde{S}} & \xrightarrow{d\tilde{\Phi}_2} & T_{\tilde{D}_2} \\
\downarrow & & \downarrow \\
\tilde{D}_2 & \xrightarrow{d\psi_k} & \tilde{D}_k
\end{array}
\]

Let us first focus on constructing the map \(\psi_k : \tilde{D}_2 \to \tilde{D}_k\). Recall from (2.5) that we have defined a reduced LLV module structure on \(H^k(X)\) by \(\rho_k : \bar{g} \to \text{gl}(H^k(X, \mathbb{Q}))\). We abuse the notation and denote the algebraic group version of this homomorphism by the same symbol
\[
\rho_k : \text{Spin}(\bar{V}, \bar{q}) \to \text{GL}(H^k(X, \mathbb{Q})).
\]

Now the desired morphism \(\psi_k\) can be constructed from this homomorphism \(\rho_k\) in the following way.

**Proposition 4.3.** There exists a unique morphism \(\psi_k : \tilde{D}_2 \to \tilde{D}_k\) with the following property. Fix any reference point \(o_2 \in \tilde{D}_2\) and let \(o_k = \psi_k(o_2)\) be its image in \(\tilde{D}_k\), so that we can identify \(\tilde{D}_2 = \text{Spin}(b_2, \mathbb{C})/P_2\) and \(\tilde{D}_k = \text{GL}(H^k(X, \mathbb{C}))/P_k\). Then the following diagram commutes.

\[
\begin{array}{ccc}
\text{Spin}(b_2, \mathbb{C}) & \xrightarrow{\rho_k} & \text{GL}(H^k(X, \mathbb{C})) \\
\downarrow & & \downarrow \\
\tilde{D}_2 & \xrightarrow{\psi_k} & \tilde{D}_k
\end{array}
\]

**Proof.** Fixing a reference point \(o_2 \in \tilde{D}_2\) amounts to fixing a Hodge structure on \(H^2(X) = \bar{V}\), and again this is equivalent to fixing the operator \(f \in \text{so}(\bar{V}, \bar{q})_\mathbb{C} = \bar{g}_\mathbb{C}\) by our discussion in Section 2. But then this operator \(f\) induces the Hodge structure on the whole cohomology \(H^*(X)\), and hence on \(H^k(X)\) (i.e., the Hodge structure on \(H^2(X)\) induces that of \(H^k(X)\)). This Hodge structure gives us a reference point on \(o_k \in \tilde{D}_k\).

The Hodge structure on \(H^2(X)\) and \(H^k(X)\) determines the Hodge structure on \(\bar{g}\) and \(\text{gl}(H^k(X))\) in the usual way. Recalling the definition of \(p_2\) and \(p_k\), we have
\[
p_2 = g^{1,1} \oplus g^{0,0}, \quad p_k = \text{gl}(H^k(X))^{k,-k} \oplus \cdots \oplus \text{gl}(H^k(X))^{0,0}.
\]

Note that reduced LLV representation \(\rho_k\) was simply defined to be the composition
\[
\rho_k : \bar{g} \subset \text{gl}(H^*(X, \mathbb{Q})) \to \text{gl}(H^k(X, \mathbb{Q})).
\]

Hence it sends the operator \(f \in \bar{g}\) in (2.23) to the Hodge operator \(f \in \text{gl}(H^k(X, \mathbb{Q}))\). This says \(\rho_k\) is a Hodge structure homomorphism, and hence sends \(p_2\) to \(p_k\). Lifting this result on the Lie algebra homomorphism \(\rho_k\) to the level of algebraic groups, we find that \(\rho_k : \text{Spin}(\bar{V}, \bar{q}) \to \text{GL}(H^k(X, \mathbb{Q}))\) sends \(p_2\) to \(p_k\). This proves \(\rho_k\) descends to a map \(\psi_k : \tilde{D}_2 \to \tilde{D}_k\). Also, by construction this map sends \(o_2\) to \(o_k\).

It remains to prove that the map \(\psi_k : \tilde{D}_2 \to \tilde{D}_k\) does not depend on the choice of a reference point \(o_2 \in \tilde{D}_2\). Let us choose a different reference point \(o'_2 = g \cdot o_2 \in \tilde{D}_2\) for \(g \in \text{Spin}(b_2, \mathbb{C})\). Letting \(F\) and \(F'\) be the Hodge filtrations on \(\bar{V}\) associated to \(o_2\) and \(o'_2\), we have \(F' = g \cdot F\). It follows \(\psi_k(o'_2) = \rho_k(g) \cdot \psi_k(o_2)\). Since \(\psi_k\) and \(\psi'_k\) are equivariant maps, this further implies \(\psi'_k = \psi_k\). □
To obtain Theorem 4.1, we need one more lemma about the contraction map. Namely, for hyper-Kähler manifolds, even the contraction homomorphism is contained in the (reduced) LLV algebra.

**Lemma 4.6.** Let \( \eta \in H^1(X, T_X) \) and \( \iota_\eta : H^*(X, \mathbb{C}) \to H^*(X, \mathbb{C}) \) be its contraction homomorphism. Then \( \iota_\eta \) is contained in the complexified reduced LLV algebra \( \mathfrak{g}_\mathbb{C} \).

**Proof.** We prove first the following identity:

\[
\iota_\eta = [L_{\iota_\eta}(\sigma), \Lambda_\sigma].
\] (4.7)

Fix a hyper-Kähler metric \( g \). The standard Hodge isomorphisms between cohomologies of holomorphic vector bundles and the set of \( C^0 \) harmonic global sections on them gives us the identifications

\[
H^1(X, T_X) = \mathcal{H}^{0,1}(X, T_X), \quad H^{p,q}(X) = \mathcal{H}^{p,q}(X), \quad H^{p-1,q+1}(X) = \mathcal{H}^{p-1,q+1}(X).
\]

Under these, we can let \( \eta \) be a global harmonic \( T_X \)-valued \((0, 1)\)-form and the contraction map \( \iota_\eta : \mathcal{H}^{p,q}(X) \to \mathcal{H}^{p-1,q+1}(X) \) to be the map wedging the \((0, 1)\)-form parts and contracting the vector field parts. Moreover, the operators \( \Lambda_{\omega_J} = \ast^{-1} L_{\omega_J} \ast \) and \( \Lambda_{\omega_K} = \ast^{-1} L_{\omega_K} \ast \) are defined as operators acting on \( C^\infty \) global differential forms. Hence the operator \( \Lambda_\sigma = \frac{1}{2}(\Lambda_{\omega_J} - \sqrt{\ast} \Lambda_{\omega_K}) \) is defined on the level of differential forms as well. Now the identity we want is about operators on differential forms. Therefore, it is enough to check the identity pointwise on \( X \).

Choose a local coordinate with \( \sigma = dz_1 \wedge dz_2 \wedge \cdots \wedge dz_{2n} \) and without loss of generality, assume \( \eta = d\bar{z}_1 \wedge \frac{\partial}{\partial z_1} \). Let \( d\bar{z}_1 \wedge dz_1 \) be a \((p, q)\)-form. Then one can compute

\[
\iota_\eta(d\bar{z}_1 \wedge dz_1) = \left( d\bar{z}_1 \wedge \frac{\partial}{\partial z_1} \right)_J(d\bar{z}_1 \wedge dz_1) = (-1)^2 d\bar{z}_1 \wedge dz_1 \wedge \left( \frac{\partial}{\partial z_1} \right). \]

To avoid using complicated indexes, here we used the notation \( v_\sigma(-) \) for the contraction \( \iota_\sigma \). To compute the right hand side, we set \( A = \frac{\partial}{\partial z_1} \wedge \frac{\partial}{\partial z_2} \wedge \cdots \wedge \frac{\partial}{\partial z_{2n}} \). Then \( \Lambda_\sigma = A_\sigma(\cdot) \). Again, computation shows

\[
[L_{\iota_\eta}(\sigma), \Lambda_\sigma](d\bar{z}_1 \wedge dz_1) = d\bar{z}_1 \wedge dz_2 \wedge (A_\sigma(d\bar{z}_1 \wedge dz_1)) - A_\sigma(d\bar{z}_1 \wedge dz_2 \wedge d\bar{z}_1 \wedge dz_1)
\]

\[
= (-1)^2 d\bar{z}_1 \wedge dz_1 \wedge dz_2 \wedge (A_\sigma(d\bar{z}_1 \wedge dz_1)) - (-1)^2 d\bar{z}_1 \wedge dz_1 \wedge (A_\sigma(dz_1 \wedge dz_1))
\]

\[
= (-1)^2 d\bar{z}_1 \wedge dz_1 \wedge (dz_2 \wedge (A_\sigma(dz_2 \wedge dz_1)) - A_\sigma(dz_2 \wedge dz_1))
\]

\[
= (-1)^2 d\bar{z}_1 \wedge dz_1 \wedge ((A_\sigma(dz_2 \wedge dz_1)) - (-1)^2 d\bar{z}_1 \wedge dz_1 \wedge (\frac{\partial}{\partial z_1} \wedge dz_1). \]

This proves the desired identity (4.7).

It follows from the identity that \( \iota_\eta \) is contained in \( \mathfrak{g}_\mathbb{C} \), by definition of the LLV algebra. Since \( \iota_\eta \) is a degree 0 operator, we can further say \( \iota_\eta \in \mathfrak{g}_0, \mathbb{C} \). Now furthermore, \( \iota_\eta \) is of bidegree \((-1, 1)\), so we have the identity \([f, \iota_\eta] = 2\sqrt{-1} \iota_\eta \). This implies \( \iota_\eta \in [\mathfrak{g}_0, \mathbb{C}, [\mathfrak{g}_0, \mathbb{C}]] = \mathfrak{g}_\mathbb{C} \). \( \square \)

**Proof of Theorem 4.1.** Let \( s \in \tilde{S} \) and \( X = X_s \) be the fiber of the family at \( s \). We fix the reference points \( \alpha_2 = \Phi_2(s) \) and \( \alpha_k = \psi_k(\alpha_2) \) of the period domains \( \tilde{D}_2 \) and \( \tilde{D}_k \), so that we can use the diagram (4.4). As it is well known (e.g. see [Voi07, Thm. 10.21]), the differential of the period map factors through the Kodaira–Spencer map

\[
d\Phi_k : T_s \tilde{S} \to H^1(X, T_X) \to T_{\Phi_k(s)} \tilde{D}_k \cong \mathfrak{gl}(H^k(X, \mathbb{C}))/\mathfrak{p}_k,
\]

with the second map being given by sending \( \eta \in H^1(X, T_X) \) to the contraction operator \((\iota_\eta)_{|H^2(X)} \). In particular, for degree 2 we have

\[
H^2(X, T_X) \to \mathfrak{g}_\mathbb{C}/\mathfrak{p}_2 = \text{Hom}(H^{2,0}(X), H^{1,1}(X)) \quad \quad \eta \mapsto (\iota_\eta)_{|H^{2,0}(X)},
\] (4.8)

which is an isomorphism by the local Torelli theorem for hyper-Kähler manifolds.

Let us consider the infinitesimal version of the diagram (4.4):

\[
\begin{array}{ccc}
\mathfrak{g}_\mathbb{C} & \xrightarrow{\rho_k} & \mathfrak{gl}(H^k(X, \mathbb{C})) \\
\phi_k & \downarrow & \downarrow \psi_k \\
H^1(X, T_X) & \xrightarrow{\iota_\eta} & \mathfrak{g}_\mathbb{C}/\mathfrak{p}_2 \xrightarrow{\delta \psi_k} \mathfrak{gl}(H^k(X, \mathbb{C}))/\mathfrak{p}_k
\end{array}
\]
We further define a left diagonal map, well-defined by Lemma 4.6, by
\[
H^1(X, T_X) \rightarrow \mathfrak{g}\mathbb{C}
\]
\[
\eta \mapsto \iota_\eta.
\]
(4.9)
The diagram certainly commutes since both (4.8) and (4.9) are defined in terms of the contraction map \(\iota_\eta\). In other words, we get
\[
d\psi_k((\iota_\eta)|_{H^2(X)}) = (\iota_\eta)|_{H^s(X)},
\]
which is precisely the commutativity of the diagram (4.2).

4.2. **Higher degree monodromy operators.** Let us now consider one-parameter degenerations \(\mathfrak{X}/\Delta\) of a compact hyper-Kähler manifold \(X\). By a one-parameter degeneration, we mean a proper flat morphism \(\pi: \mathfrak{X} \rightarrow \Delta\) over the unit disk \(\Delta\) such that the restriction \(\pi|_{\Delta^*}: \mathfrak{X}^* \rightarrow \Delta^*\) to the punctured disk \(\Delta^*\) is a smooth family. We assume that a general fiber \(X_s\) is isomorphic to \(X\) (a hyper-Kähler in a given deformation class). By Ehresmann’s theorem, the family over the punctured disk can be locally trivialized, and one gets a monodromy operator \(T_k\) for each cohomological degree.

**Definition 4.10.** Let \(\pi: \mathfrak{X} \rightarrow \Delta\) be a one-parameter degeneration of \(2n\)-dimensional (hyper-)Kähler manifolds, and \(s_0 \in \Delta\) be a base point. For \(k \in \{0, \ldots, 4n\}\), let
\[
\pi_1(\Delta^*, s_0) \rightarrow \text{GL}(H^k(X))
\]
be the monodromy representation associated to the local system \(R^k(\pi|_{\Delta^*})*\) on \(\Delta^*\). The \(k\)-th monodromy
\[
T_k \in \text{GL}(H^k(X))
\]
is the image of the generator \(1 \in \pi_1(\Delta^*, s_0) \cong \mathbb{Z}\) by this homomorphism. Since each \(T_k\) is a quasi-unipotent operator for \(0 \leq k \leq 4n\) (e.g. [Sch73]), we define the \(k\)-th log monodromy operator by
\[
N_k = \frac{1}{m} \log(T_k)^m \in \text{End}(H^k(X)),
\]
where \(m \in \mathbb{Z}_{>0}\) is such that \((T_k)^m\) is unipotent.

For hyper-Kähler manifolds the second cohomology controls everything. The following result says that the second monodromy operator \(N_2\) determines all the monodromy operators \(N_k\) by means of the LLV algebra. To make this precise, it is easy to see that \(N_2\) respects the Beauville–Bogomolov form on \(H^2(X)\), so that \(N_2 \in \mathfrak{g} = \mathfrak{so}(\tilde{V}, \tilde{q})\). On the other hand, we recall again (2.5) there was the reduced LLV representation \(\rho_k: \mathfrak{g} \rightarrow \mathfrak{gl}(H^k(X))\). Hence we can consider the operator \(\rho_k(N_2)\). It was already observed by Soldatenkov [Sol18, Prop. 3.4] that this is precisely the \(k\)-th log monodromy operator \(N_k\). We recover this result by using the higher degree period maps and domains discussed in §4.1.

**Theorem 4.12.** With notations as above, the \(k\)-th log monodromy is determined by the second log monodromy by the relation
\[
N_k = \rho_k(N_2).
\]

**Proof.** Since we are interested in the log monodromies, base changing the original family by \(t \mapsto t^m\) does not affect our statement. Thus, we may assume all the monodromies \(T_k\) are unipotent. Then by the nilpotent orbit theorem, the derivative of the period maps can be expressed by
\[
\frac{d\tilde{\Phi}_2}{dt}(t) = \frac{N_2}{2\pi \sqrt{-1}} t^{-1} + (\text{holomorphic part}) \quad \in \mathfrak{g}\mathbb{C}/\mathfrak{p}_2;
\]
\[
\frac{d\tilde{\Phi}_k}{dt}(t) = \frac{N_k}{2\pi \sqrt{-1}} t^{-1} + (\text{holomorphic part}) \quad \in \mathfrak{gl}(H^k(X, \mathbb{C}))/\mathfrak{p}_k.
\]
Applying Theorem 4.1 and the fact that \(\psi_k\) is derived from \(\rho_k\) by Proposition 4.3, we have an identity
\[
N_k = \rho_k(N_2) \pmod{\mathfrak{p}_k}.
\]
However, note that \(N_2 \in \mathfrak{g}^{-1,1}\) and \(N_2 \in \mathfrak{gl}(H^k(X))^{-1,1}\). Since \(\rho_k\) is a Hodge structure morphism, we also get \(\rho_k(N_2) \in \mathfrak{gl}(H^k(X))^{-1,1}\). Since \(\mathfrak{p}_k = \mathfrak{gl}(H^k(X))^{k,-k} \oplus \cdots \oplus \mathfrak{gl}(H^k(X))^{0,0}\), we conclude that in fact
\[
N_k = \rho_k(N_2).
\]
\(\square\)
5. Nagai’s conjecture and the Looijenga–Lunts–Verbitsky decomposition

In this section, we give a representation theoretic interpretation to Nagai’s conjecture (1.9) on the index of nilpotency of the higher degree monodromy operators $N_k$. The representation theoretic approach to (1.9) appears in the original work of Nagai [Nag08], and more recently Soldatenkov [Sol18]. Nonetheless, our representation theoretic criterion (Proposition 1.11 = Theorem 5.4 below) seems new, and presents Nagai’s conjecture as a natural next condition satisfied by the cohomology of hyper-Kähler manifolds (compare Proposition 2.34).

As before, let $X$ be a compact hyper-Kähler manifold, $\bar{V} = H^2(X)$ its second cohomology endowed with the Beauville–Bogomolov form $\bar{q}$, and $(V, q)$ the Mukai completion of $(\bar{V}, \bar{q})$. The LLV algebra $\mathfrak{g} \cong \mathfrak{so}(V, q)$ acts on the cohomology $H^*(X)$. All this data is diffeomorphism invariant. Consider now a one-parameter degeneration $X/\Delta$ of compact hyper-Kähler manifolds of dimension $2n$. For each $0 \leq k \leq 4n$, let $T_k$ be the monodromy operator on the $k$-th cohomology of the degeneration. The monodromy $T_k$ lives in the group $\text{GL}(H^k(X))$. By the monodromy theorem (see, e.g., [Sch73, Thm 6.1]), all $T_k$ are quasi-unipotent operators.

We define the log monodromies by

$$N_k = \frac{1}{m} \log(T_k)^m \in \text{End}(H^k(X)),$$

where $m$ is the smallest positive integer (depending on $k$) such that $(T_k)^m$ becomes unipotent. Note that the second monodromy $T_2$ respects the Beauville–Bogomolov form $\bar{q}$. This implies $T_2$ is further contained in $\text{SO}(\bar{V}, \bar{q})$, and hence $N_2$ is contained in the reduced LLV algebra $\bar{\mathfrak{g}} \cong \mathfrak{so}(\bar{V}, \bar{q})$. We would like to distinguish the second log monodromy $N_2$ as an element in the endomorphism ring $\text{End}(H^2(X))$ and as an element in the reduced LLV algebra $\bar{\mathfrak{g}}$. For the latter, we will rather denote it by

$$N \in \bar{\mathfrak{g}} \cong \mathfrak{so}(\bar{V}, \bar{q}).$$

For each $k$, we can associate to $N_k$ its index of nilpotency $\nu_k$, i.e., $\nu_k$ is the largest nonnegative integer with $(N_k)^{\nu_k} \neq 0$. Nagai [Nag08] observed that the following relation between the nilpotency indexes of even cohomologies:

$$\nu_{2k} = k \cdot \nu_2 \quad \text{for} \quad k = 1, \ldots, n.$$  (1.9 (restated))

holds at least in some cases (e.g., for degenerations of $K3^{[n]}$ type). Indeed, one of our main results (Theorem 1.10) is that this is true for the known examples of hyper-Kähler manifolds. For now, in this section, we recast the condition (1.9) in terms of the LLV decomposition of the cohomology $H^*(X)$. Since (1.9) depends only on the even cohomology, let us write the LLV decomposition on the even cohomology part:

$$H^*_{\text{even}}(X) \cong \bigoplus_{\mu \in S} V_\mu^{\oplus m_\mu},$$

(5.3)

where $\mu = (\mu_0, \cdots, \mu_r)$ indicates a dominant integral weight of $\mathfrak{g}$ and $V_\mu$ its associated highest weight module.

With these preliminaries, we can state the main result of the section.

**Theorem 5.4.** Let $X$ be a compact hyper-Kähler manifold with $b_2(X) \geq 4$ and $X/\Delta$ a one-parameter degeneration of $X$. Let $\nu_2$ and $\nu_{2k}$ be the nilpotency index of the second and $2k$-th log monodromies of the degeneration. Then Nagai’s conjecture (1.9) holds if and only if every irreducible $\mathfrak{g}$-module component $V_\mu$ appearing in the LLV decomposition of the even cohomology (5.3) satisfies the inequality

$$\mu_0 + \mu_1 + |\mu_2| \leq n.$$  (5.5)

**Remark 5.6.** The absolute value on $\mu_2$ in the inequality is to cover the case $b_2(X) = 4$ (i.e., if $b_2(X) \geq 5$, $\mu_2 \geq 0$). Namely, the representation theory of simple Lie algebra of type $D_r+1$ says that any dominant integral weight $\mu = (\mu_0, \cdots, \mu_r)$ satisfies $\mu_0 \geq \cdots \geq |\mu_{r-1}| \geq |\mu_r| \geq 0$. If $b_2(X) = 4$ and thus $r = 2$, some care is needed for $\mu_2 < 0$ (otherwise $\mu_2 \geq 0$; similar considerations apply for the $B_{r+1}$ case).

**Remark 5.7.** The Looijenga–Lunts–Verbitsky decomposition of the cohomology only depends on the $C^\infty$ structure of $X$. Therefore, Theorem 5.4 tells us that Nagai’s conjecture is about the $C^\infty$ structure of $X$. More precisely, we prove that Nagai’s conjecture holds for type I and III degenerations without any projectivity assumption. Then, the main content of Theorem 5.4 is to say that Nagai’s conjecture holds for type II degenerations if and only if the inequality (5.5) holds (which only depends on the $C^\infty$ structure of $X$). In particular, this means that the arguments of Nagai [Nag08], who checked that (1.9) holds for some
degenerations of Kum$_n$ type, give the Nagai’s conjecture also for Kum$_n$ (in addition to K3$^{[n]}$). Equivalently, $\mu_0 + \mu_1 + \mu_2 \leq n$ for hyper-Kähler manifolds of Kum$_n$ type. Later (Theorem 6.1), we will see that the stronger inequality $\mu_0 + \mu_1 + \mu_2 + \mu_3 \leq n$ holds for Kum$_n$.

The rest of this section will be devoted to the proof of Theorem 5.4. We divide the proof into three cases depending on the Type of the degeneration. Specifically, recall the Type of the degeneration is I, II and III of the degeneration depending on the nilpotency index $\nu_2 \in \{0, 1, 2\}$ of the second log monodromy. Under a projectivity assumption, the Type I and III were established in [KLSV17]. From our perspective, the Type I is trivial, as $N_{2k}$ is determined (see Section 4) via the LLV decomposition from $N_{2k}(= 0$ for Type I). The argument for Type III is similar to that in [KLSV17] (essentially the Verbitsky component $V_{(n)}$ is always present in the LLV decomposition). Finally, the Type II case requires a more delicate representation theoretic argument.

5.1. Type I and III degenerations. In Section 4 we have discussed the interplay between the LLV decomposition and the period map. In particular, we have seen that in the case of one-parameter degenerations, the monodromy operator $N \in \mathfrak{g}$ (cf. (5.2)) determines (Theorem 4.12) all the monodromy operators $N_k$ by

$$N_k = \rho_k(N)$$

where $\rho_k : \mathfrak{g} \to \text{End}(H^k(X))$ is the representation induced from the LLV decomposition (it is essentially, the restriction by degree of the LLV decomposition). As an immediate consequence we obtain the Type I and III cases of Nagai’s conjecture without any further restrictions.

**Proposition 5.8.** Nagai’s conjecture (1.9) holds for type I and III degenerations of compact hyper-Kähler manifolds.

**Proof.** Type I degeneration is equivalent to $\nu_2 = 0$, i.e. $N = 0$. Since $N_k = \rho_k(N)$, we conclude $N_k = 0$ for all $k$ (compare [KLSV17, Cor. 3.2]). In particular, $\nu_{2k} = 0$ as needed.

For Type III degeneration ($\nu_2 = 2$), on one hand, we have the general bound on the index on nilpotency on $H^{2n}$, i.e. $\nu_{2k} \leq 2k$. Conversely, we recall that the LLV decomposition of $H^*(X)$ always contains the Verbitsky component $V_{(n)}$. From Appendix B, Verbitsky component splits as $\text{Sym}^k \tilde{V} \subset H^{2k}(X)$ as a $\mathfrak{g}$-module. Hence we have a $\mathfrak{g}$-module decomposition

$$H^{2k}(X) = \text{Sym}^k \tilde{V} \oplus V'_{2k},$$

or equivalently $\rho_{2k}$ splits as $\text{Sym}^k \rho_2 \oplus \rho'_{2k}$, where $\text{Sym}^k \rho_2 : \mathfrak{g} \to \text{End}(\text{Sym}^k \tilde{V})$ and $\rho'_{2k}$ is some residual representation. Since $N_{2k} = \rho_{2k}(N)$, and the nilpotency index on $\text{Sym}^k \rho_2$ is $2k$ (cf. Lemma 5.9), the claim follows.

The following lemma is standard, for completeness, we include the proof.

**Lemma 5.9.** Let $\text{Sym}^k \rho_2 : \mathfrak{g} \to \text{End}(\text{Sym}^k \tilde{V})$ be the $\mathfrak{g}$-module structure on $\text{Sym}^k \tilde{V}$. Then $\text{Sym}^k \rho_2(N)$ has nilpotency index $k \cdot \nu_2$.

**Proof.** The operator $\text{Sym}^k \rho_2(N)$ acts on $\text{Sym}^k \tilde{V}$ as follows:

$$(\text{Sym}^k \rho_2(N))(x_1 \cdots x_k) = \sum_{i=1}^{k} x_1 \cdots (N_2 x_i) \cdots x_k.$$  

Recall by definition of the nilpotency index $\nu_2$, we have $(N_2)^{\nu_2+1} = 0$ but $(N_2)^{\nu_2} \neq 0$. One computes

$$(\text{Sym}^k \rho_2(N))^{k\nu_2}(x^k) = (\text{const.})((N_2)^{\nu_2}x)^k, \quad (\text{Sym}^k \rho_2(N))^{k\nu_2+1} = 0,$$

which establishes the claim.

5.2. Type II degeneration. Consider the reduced LLV decomposition the $2k$-th cohomology

$$H^{2k}(X) \cong \bigoplus_{\lambda} \tilde{V}_{\lambda}^{\text{sym}_{\lambda}}.$$  

(5.10)

Here $\lambda = (\lambda_1, \cdots, \lambda_r)$ denotes a dominant integral weight of $\mathfrak{g}$ and $\tilde{V}_{\lambda}$ denotes a highest $\mathfrak{g}$-module of weight $\lambda$. Proposition 2.35 tells us that every $\lambda$ in this decomposition has integer coefficients $\lambda_i$. For each of such components $\tilde{V}_{\lambda}$’s, we can in fact compute the nilpotency index of the log monodromy $N_{2k}$ on this component. This is the content of the next lemma, which is the core computation used in the proof of Theorem 5.4.
Lemma 5.11. Assume $b_2(X) \geq 4$. Let $\lambda = (\lambda_1, \cdots, \lambda_r)$ be a dominant integral weight of $\mathfrak{g}$ with $\lambda_i \in \mathbb{Z}$ and $\rho_{\lambda}: \mathfrak{g} \to \text{End}(\bar{V}_{\lambda})$ the highest $\mathfrak{g}$-module associated to it.

(i) If $\nu_2 = 1$, then $\rho_{\lambda}(N)$ has nilpotency index $\lambda_1 + |\lambda_2|$.
(ii) If $\nu_2 = 2$, then $\rho_{\lambda}(N)$ has nilpotency index $2\lambda_1$.

Note that the lemma holds under the assumption $b_2(X) \geq 4$. This is the reason we have the same assumption in Theorem 5.4. The case $b_2(X) = 3$ is exceptional because the rank of the Lie algebra $\mathfrak{g}$ becomes too small. This exceptional case will be managed separately later on. Also, we note that the proof of this lemma is not purely representation theoretic. The fact that $N$ is obtained from a degeneration of Hodge structure, and hence associated to a limit mixed Hodge structure of the Hodge structure $\bar{V}$ of K3 type, will be crucially used. For an arbitrary choice of an element $N \in \mathfrak{g}$, the lemma would not hold.

The proof of Lemma 5.11 is quite lengthy, so we would like to devote the rest of this subsection for its proof. The proof of Theorem 5.4 is then completed in §5.3. Before getting into the proof, note that the statement of the lemma does not depend on the base field. Hence, it is enough to prove the lemma over $\mathbb{C}$. For simplicity, let us omit the base change index $\mathbb{C}$ and assume everything is complexified from now on.

The first step is to give a normalization of the monodromy action on $\bar{V} = H^2(X)$. Since we are working over $\mathbb{C}$, we can assume that the quadratic space $(\bar{V}, \bar{q})$ has one of the following standard form:

$$
\begin{pmatrix}
0 & \text{id}_{r \times r} & 0 \\
\text{id}_{r \times r} & 0 & 0 \\
0 & 0 & 1
\end{pmatrix}
\quad \text{or} \quad
\begin{pmatrix}
0 & \text{id}_{r \times r} \\
\text{id}_{r \times r} & 0
\end{pmatrix}
$$

(5.12)

depending on the parity of the dimension of $\bar{V}$. The content of the following proposition is to say that both $N$ and $\bar{q}$ can be suitably normalized.

Lemma 5.13. Assume $b_2(X) = \dim \bar{V} \geq 4$. Let $N_2 \in \text{End}(\bar{V})$ be the second log monodromy and $\nu_2$ its nilpotency index.

(i) If $\nu_2 = 1$, then $\dim(\text{im} N_2) = 2$. Moreover, there exists a basis

$$\{e_1, \cdots, e_r, e'_1, \cdots, e'_r\}$$

of $\bar{V}$ with respect to it has a matrix form (5.12), and

$$N_2 \left( \sum_{i=1}^{r} a_i e_i + \sum_{i=1}^{r} a'_i e'_i \right) = a_2 e'_1 + a_1 e'_2. \quad (5.14)$$

(ii) If $\nu_2 = 2$, then $\dim(\text{im} N_2) = 2$ and $\dim(\text{im}(N_2)^2) = 1$. Moreover, there exists a basis

$$\{e_1, \cdots, e_r, e'_1, \cdots, e'_r\}$$

of $\bar{V}$ with respect to it has a matrix form (5.12), and

$$N_2 \left( \sum_{i=1}^{r} a_i e_i + \sum_{i=1}^{r} a'_i e'_i \right) = a_1 e_2 - (a_2 + a'_2) e'_1 + a_1 e'_2. \quad (5.15)$$

Proof. Our arguments follow closely [SS17, Prop 4.1] (they go back to the study of degenerations of K3 surface, e.g. in [FS86] even a normalization over $\mathbb{Z}$ is given). For completeness and notational consistency, we give a proof. (Although, we typically distinguish between $N$ and $N_2$, here we simply write $N$ for $N_2$.)

Assume first that we have a Type I degeneration. The one-parameter degeneration produces a limit mixed Hodge structure $\bar{V}_{\text{lim}}$. It is a degeneration of the second cohomology $\bar{V}$ of K3 type. The nilpotency index is $\nu_2 = 1$, so we have the monodromy weight filtration

$$0 \subset W_1 \subset W_2 \subset W_3 = \bar{V}_{\text{lim}}$$

with $W_1 = \text{im} N$ and $W_2 = \ker N$. Since it is a degeneration of a K3 type Hodge structure, there is only one possibility of the Hodge diamond of $\bar{V}_{\text{lim}}$ as in Table 3. From it, we deduce $\dim W_1 = 2$ and $\dim W_2 = b_2(X) - 2$. This proves $\dim(\text{im} N) = 2$. Next, we choose two elements $x, y \in \bar{V}$ in as follows:

(i) Choose any $x \notin \ker N$. Choose any $y \in x^\perp \setminus (Nx)^\perp$. Since $Nx \in \ker N$, $x$ and $Nx$ are linearly independent.

(ii) Adjust $y$ so that $\bar{q}|_{\{y, Nx\}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. This is possible because $\bar{q}(Nx) = -(x, N^2 x) = 0.$
Assume first that we are in the Type II case. We get
\[(\rho(N))^2(x_1 \wedge \cdots \wedge x_i) = 2 \sum N_2 x_1 \wedge N_2 x_2 \wedge x_3 \wedge \cdots \wedge x_i,
\]
\[(\rho(N))^3(x_1 \wedge \cdots \wedge x_i) = 0.
\]
This proves the nilpotency index of \(\rho(N)\) is at most 2. Using the basis of Lemma 5.13, we get
\[(\rho(N))^2(e_1 \wedge \cdots \wedge e_i) = 2 e'_1 \wedge e'_2 \wedge e_3 \wedge \cdots \wedge e_i \neq 0.
\]
This proves the nilpotency index of \(\rho(N)\) is precisely 2.

In the Type III case we have \(\dim(\text{im}(N_2)) = 2\), \(\dim(\text{im}(N_2^2)) = 1\), and \((N_2)^3 = 0\). Thus
\[(\rho(N))^2(x_1 \wedge \cdots \wedge x_i) = \sum (N_2)^2 x_1 \wedge x_2 \wedge \cdots \wedge x_i + 2 \sum N_2 x_1 \wedge N_2 x_2 \wedge x_3 \wedge \cdots \wedge x_i,
\]
\[(\rho(N))^3(x_1 \wedge \cdots \wedge x_i) = 3 \sum (N_2)^2 x_1 \wedge N_2 x_2 \wedge x_3 \wedge \cdots \wedge x_i.
\]
Now using the preferred basis, we can further compute
\[(\rho(N))^2(e_1 \wedge \cdots \wedge e_i) = 2 e'_1 \wedge e'_2 \wedge e_3 \wedge \cdots \wedge e_i \neq 0,
\]
\[(\rho(N))^3 = 0.
\]
We define \(e_1 = x, e_2 = y, e'_1 = -Ny, \) and \(e'_2 = Nx\). By construction, the intersection pairing on the subspace spanned by \(\{e_1, e_2, e'_1, e'_2\}\) is \(U^{\otimes 2}\) as needed. For the orthogonal space \(\langle e_1, e_2, e'_1, e'_2 \rangle^\perp\) we choose a basis \(\{e_3, \cdots, e_r, e'_3, \cdots, e'_r, e_{r+1}\}\) as needed in needed in the normal form (5.12) (since working over \(\mathbb{C}\), this can be accomplished). Finally note that since \(e_3, \cdots, e_r, e'_3, \cdots, e'_r, e_{r+1}\) are all perpendicular to \(e'_1\) and \(e'_2\), they are perpendicular to \(\text{im}(N)\), and thus contained in \(\ker N = (\text{im}(N))^\perp\). The formula (5.14) for \(N\) follows.

The Type III case is similar (and standard). We omit the details. □

Once a choice of basis as in the lemma above has been made, we can further adjust the Cartan subalgebra and simple roots of \(\tilde{\mathfrak{g}}\) so that it becomes compatible with this choice of basis.

**Lemma 5.16.** We can choose a Cartan subalgebra \(\mathfrak{h} \subset \tilde{\mathfrak{g}}\) and simple roots of \(\tilde{\mathfrak{g}}\) so that the basis elements \(e_1, \cdots, e_r, e'_1, \cdots, e'_r\) (and \(e_{r+1}\)) in Lemma 5.13 are the weight vectors associated to the weights \(\varepsilon_1, \cdots, \varepsilon_r, -\varepsilon_1, \cdots, -\varepsilon_r\) (and 0) of the standard \(\tilde{\mathfrak{g}}\)-module \(\bar{V}\).

The normalizations above allow us to compute explicitly the nilpotency index as stated in Lemma 5.11. First, we have the following special case of Lemma 5.11.

**Lemma 5.17.** Assume \(b_2(X) = \dim \bar{V} \geq 4\). Consider the \(\tilde{\mathfrak{g}}\)-module \(\rho: \tilde{\mathfrak{g}} \to \text{End}(\wedge^i \bar{V})\) with \(2 \leq i \leq r\).

(i) If \(\nu_2 = 1\), then \(\rho(N)\) has nilpotency index 2.

(ii) If \(\nu_2 = 2\), then \(\rho(N)\) also has nilpotency index 2.

**Proof.** Assume \(\nu_2 = 1\). We have an explicit computation
\[(\rho(N))(x_1 \wedge \cdots \wedge x_i) = \sum_{j=1}^i x_1 \wedge \cdots \wedge N_2 x_j \wedge \cdots \wedge x_i.
\]
Assume first that we are in the Type II case. We get
\[(\rho(N))^2(x_1 \wedge \cdots \wedge x_i) = 2 \sum N_2 x_1 \wedge N_2 x_2 \wedge x_3 \wedge \cdots \wedge x_i,
\]
\[(\rho(N))^3(x_1 \wedge \cdots \wedge x_i) = 0.
\]
This proves the nilpotency index of \(\rho(N)\) is at most 2. Using the basis of Lemma 5.13, we get
\[(\rho(N))^2(e_1 \wedge \cdots \wedge e_i) = 2 e'_1 \wedge e'_2 \wedge e_3 \wedge \cdots \wedge e_i \neq 0.
\]
This proves the nilpotency index of \(\rho(N)\) is precisely 2.

In the Type III case we have \(\dim(\text{im}(N_2)) = 2\), \(\dim(\text{im}(N_2^2)) = 1\), and \((N_2)^3 = 0\). Thus
\[(\rho(N))^2(x_1 \wedge \cdots \wedge x_i) = \sum (N_2)^2 x_1 \wedge x_2 \wedge \cdots \wedge x_i + 2 \sum N_2 x_1 \wedge N_2 x_2 \wedge x_3 \wedge \cdots \wedge x_i,
\]
\[(\rho(N))^3(x_1 \wedge \cdots \wedge x_i) = 3 \sum (N_2)^2 x_1 \wedge N_2 x_2 \wedge x_3 \wedge \cdots \wedge x_i.
\]
Now using the preferred basis, we can further compute
\[(\rho(N))^2(e_1 \wedge \cdots \wedge e_i) = 2 e'_1 \wedge e'_2 \wedge e_3 \wedge \cdots \wedge e_i \neq 0,
\]
\[(\rho(N))^3 = 0.
\]
(We have been unable to see the last identity without working with a suitable basis). The lemma follows. □
One subtlety that needs attention is that $\wedge^r \tilde{V}$ is not an irreducible $\tilde{g}$-module for $b_2(X) = \dim \tilde{V}$ even. In fact, in that case, it holds $\wedge^r \tilde{V} = \tilde{V}_{2\varpi_{r-1}} \oplus \tilde{V}_{2\varpi_r}$. See Appendix A. With this in mind, we complete the proof of Lemma 5.11.

**Proof of Lemma 5.11.** Set

$$a_i = \lambda_i - \lambda_{i+1} \quad \text{for} \quad 1 \leq i \leq r - 2, \quad a_{r-1} = \lambda_{r-1} - |\lambda_r|, \quad a_r = |\lambda_r|.$$ 

Consider a $\tilde{g}$-module

$$W = \text{Sym}^{a_1} \tilde{V} \otimes \text{Sym}^{a_2}(\wedge^2 \tilde{V}) \otimes \cdots \otimes \text{Sym}^{a_r}(\wedge^r \tilde{V}).$$

The highest weight of this module becomes exactly $\lambda$ (there are two highest weights when $b_2(X)$ is even and $\lambda_r \neq 0$, the other one being $(\lambda_1, \ldots, \lambda_{r-1}, -\lambda_r)$). Hence $\tilde{V}_\lambda$ should be contained in $W$. Using Lemma 5.17, this proves the nilpotency index of $\rho_\lambda(N)$ is at most $a_1 + 2a_2 + \cdots + 2a_r = \lambda_1 + |\lambda_2|$ for $n_2 = 1$, and $2a_1 + \cdots + 2a_r = 2\lambda_1$ for $n_2 = 2$.

To prove the nilpotency index is precisely the desired value, we need Lemma 5.13 with Lemma 5.16. Using these results, one can see that $\wedge^i \tilde{V}$ has a unique highest weight vector $e_1 \wedge \cdots \wedge e_i$ up to scalar (there are two when $b_2(X)$ is even and $i = r$, the other one being $e_1 \wedge \cdots \wedge e_{r-1} \wedge e_i$). Hence $W$ has a unique (two) highest weight vector, up to scalar,

$$x := e_1^{a_1} \otimes (e_1 \wedge e_2)^{a_2} \otimes \cdots \otimes (e_1 \wedge \cdots \wedge e_r)^{a_r},$$

and this $x$ is contained in $\tilde{V}_\lambda$.

Assume $n_2 = 1$. Then the computations in the proof of Lemma 5.17 shows

$$(\rho(N))^{a_1 + 2a_2 + \cdots + 2a_r}(x) = (e_1^{a_1}) \otimes (2e_1 \wedge e_2)^{a_2} \otimes \cdots \otimes (2e_1 \wedge \cdots \wedge e_r)^{a_r} \neq 0.$$ 

Assume $n_2 = 2$. Then again, computations in Lemma 5.17 shows

$$(\rho(N))^{2a_1 + 2a_2 + \cdots + 2a_r}(x) = (-2e_1)^{a_1} \otimes (2e_1 \wedge e_2)^{a_2} \otimes \cdots \otimes (2e_1 \wedge \cdots \wedge e_r)^{a_r} \neq 0.$$ 

This proves the nilpotency indexes are precisely as stated. \hfill \Box

### 5.3. Completion of the proof of Theorem 5.4.

Now that we know Lemma 5.11 holds, we can compute the nilpotency index $n_{2k}$ explicitly.

**Proposition 5.18.** Assume $b_2(X) \geq 4$ and $n_2 = 1$. Let $H^{2k}(X) = \bigoplus_{\lambda \in S} \tilde{V}_\lambda^{\oplus n_\lambda}$ be a $\tilde{g}$-module irreducible decomposition of the $2k$-th cohomology. Then

$$n_{2k} = \max\{\lambda_1 + |\lambda_2| : \lambda = (\lambda_1, \ldots, \lambda_r) \in S\}.$$ 

**Proof.** The representation $\rho_{2k} : \tilde{g} \to \text{End}(H^{2k}(X))$ decomposes into

$$\rho_{2k} : \tilde{g} \to \bigoplus_{\lambda \in S} \text{End}(\tilde{V}_\lambda)^{\oplus n_\lambda} \subset \text{End}(H^{2k}(X)).$$

Hence $\rho_{2k}(N)$ is the direct sum of each $\rho_\lambda(N)$, and its nilpotency index is the maximum of those of $\rho_\lambda(N)$. Thus the statement follows from Lemma 5.11. \hfill \Box

By Lemma 5.9, if $n_2 = 1$ then we always have $n_{2k} \geq k$. Thus, it is enough to show every irreducible $\tilde{g}$-module component $\tilde{V}_\lambda$ of $H^{2k}(X)$ satisfies $\lambda_1 + |\lambda_2| \leq k$.

**Corollary 5.19.** Assume $b_2(X) \geq 4$ and $n_2 = 1$. Then $n_{2k} = k$ for all $0 \leq k \leq n$ if and only if every highest $\tilde{g}$-weight appearing in (5.10) satisfies the inequality $\lambda_1 + |\lambda_2| \leq k$ for all $0 \leq k \leq n$. \hfill \Box

The final step now is to lift the condition $\lambda_1 + |\lambda_2| \leq k$ (in terms of the $\tilde{g}$-module structure on $H^k(X)$) to a condition in terms of the $g$-module structure on $H^k_{\text{even}}(X)$. Recall that the $\tilde{g}$-module structure on $H^{2k}(X)$ was induced from a more rigid $g$-module structure on $H^k_{\text{even}}(X)$. The $g$-module irreducible decomposition of the full even cohomology was

$$H^*_\text{even}(X) \cong \bigoplus_{\mu \in S} V_\mu^{\oplus n_\mu},$$

(5.3 (rephrased))

where $\mu = (\mu_0, \ldots, \mu_r)$ indicates a dominant integral weight of $g$ and $V_\mu$ indicates the associated $g$-module.

Let us start from the lifting of $\tilde{g}$-module structure to the $g_0$-module structure. Recall the definition $g_0 = \tilde{g} \oplus R h$ in (2.4). Assume $0 \leq k \leq n$. The $g_0$-module $\tilde{V}_\lambda$ contained in $H^{2k}(X)$ can be think of an irreducible $g_0$-module of highest weight $(k - n)\epsilon_0 + \lambda$ contained in $H^*_\text{even}(X)$. This is because the operator
\( h = \varepsilon_0^2 \) acts on \( H^{2k}(X) \) by the multiplication \( 2k - 2n \), whence giving us the coefficient \((k - n)\varepsilon_0\). We abuse our notation and write this \( \mathfrak{g}_0 \)-module as

\[
\tilde{V}_{(k-n)\varepsilon_0 + \lambda}.
\]

Note that the Cartan subalgebra \( \mathfrak{h} \subset \mathfrak{g}_0 \) of \( \mathfrak{g} \) and \( \mathfrak{g}_0 \) are exactly the same. The difference between their representation theory comes from their Weyl group. The Weyl group \( \mathfrak{W}_0 \) of \( \mathfrak{g}_0 \) is strictly smaller than the Weyl group \( \mathfrak{W} \) of \( \mathfrak{g} \); the Weyl group \( \mathfrak{W}_0 \) loses all the symmetries coming from the weight \( \varepsilon_0 \). This explains why \( V_\mu \) decomposes further as a \( \mathfrak{g}_0 \)-module.

Fix a Cartan subalgebra \( \mathfrak{h} \subset \mathfrak{g}_0 \) of \( \mathfrak{g} \). The weights of \( \mathfrak{g} \) live in the space \( \mathfrak{h}_R^* \). We define the weight polytope \( WP(V_\mu) \) of \( V_\mu \) as the smallest convex hull in \( \mathfrak{h}_R^* \) containing all the weights of \( V_\mu \). The following simple lemma about the weight polytope will be useful.

**Lemma 5.20.** Let us define a subset of \( \mathfrak{h}_R^* \) by

\[
K = \{ \sum_{i=0}^{r} t_i \varepsilon_i \in \mathfrak{h}_R^* : t_i \in \mathbb{R} \text{ for } 0 \leq i \leq r, \quad |t_0| + |t_1| + |t_2| \leq n \}.
\]

If a dominant integral weight \( \mu \) of \( \mathfrak{g} \) is contained in \( K \), then the whole weight polytope \( WP(V_\mu) \) is contained in \( K \).

**Proof.** Note that a dominant integral weight \( \mu = \sum_{i=0}^{r} \mu_i \varepsilon_i \) satisfies \( \mu_0 \geq \cdots \geq \mu_{r-1} \geq |\mu_r| \geq 0 \). Thus, \(|\mu_0| + |\mu_1| + |\mu_2| \leq n \) implies \(|\mu_i| + |\mu_j| + |\mu_k| \leq n \) for all different \( i, j, k \). Now, a Weyl group action \( w \in \mathfrak{W} \) of type BD acts on \( \mu \) by permutation of coefficients \( \mu_i \) and changing their signs. Hence the sum of absolute value of the first three coefficients of \( w.\mu \) is always \(|\mu_i| + |\mu_j| + |\mu_k| \leq n \). This proves all the vertices of \( WP(V_\mu) \) is contained in \( K \). Since the weight polytope \( WP(V_\mu) \) is a convex hull generated by its vertices and \( K \) is a convex set, we conclude the statement. \( \square \)

We now conclude the proof of Theorem 5.4.

**Proof of Theorem 5.4.** Assume \( \mu_0 + \mu_1 + |\mu_2| \leq n \) for all \( \mu \in S \). Consider any \( \tilde{V}_{\lambda} \subset H^{2k}(X) \). We lift it to a \( \mathfrak{g}_0 \)-module \( V_{(k-n)\varepsilon_0 + \lambda} \subset H^*(X) \). Then there exists a unique irreducible \( \mathfrak{g} \)-submodule \( V_\mu \subset H^*(X) \) containing \( V_{(k-n)\varepsilon_0 + \lambda} \). For such \( \mu \), we certainly have \((k - n)\varepsilon_0 + \lambda \in WP(V_\mu)\). Now Lemma 5.20 says \((n - k) + \lambda_1 + |\lambda_2| \leq n \). Hence \( \lambda_1 + |\lambda_2| \leq k \), and now Nagai’s conjecture follows from Corollary 5.19.

Conversely, assume there exists \( \mu \in S \) with \( \mu_0 + \mu_1 + |\mu_2| > n \). Define a dominant integral weight \( \lambda \) of \( \mathfrak{g} \) by

\[
\lambda = \begin{cases} 
\mu_1 \varepsilon_1 + \cdots + \mu_r \varepsilon_r & \text{if } b_2(X) \text{ is odd}, \\
\mu_1 \varepsilon_1 + \cdots + \mu_{r-1} \varepsilon_{r-1} - \mu_r \varepsilon_r & \text{if } b_2(X) \text{ is even}.
\end{cases}
\]

Consider a dominant integral \( \mathfrak{g}_0 \)-weight \( -\mu_0 \varepsilon_0 + \lambda \). It is also a \( \mathfrak{g} \)-weight since the weight lattices of \( \mathfrak{g}_0 \) and \( \mathfrak{g} \) are the same. Let us define the Weyl group action \( w \in \mathfrak{W} \) as follows: if \( b_2(X) \) is odd then \( w \) changes the sign of \( \varepsilon_0 \), and if \( b_2(X) \) is even then \( w \) changes the sign of both \( \varepsilon_0 \) and \( \varepsilon_r \). Regardless of the parity of \( b_2(X) \), we always have \(-\mu_0 \varepsilon_0 + \lambda = w.\mu \). Since \( \mu \in WP(V_\mu) \), we have \(-\mu_0 \varepsilon_0 + \lambda \in WP(V_\mu) \) as one of the vertices. This forces \( V_{-\mu_0 \varepsilon_0 + \lambda} \subset V_\mu \subset H^*(X) \) as \( \mathfrak{g}_0 \)-submodules. But then by the discussion above, we have \( \tilde{V}_{\lambda} \subset H^{2k}(X) \) for \( k = -\mu_0 + n \) with the property \( \lambda_1 + |\lambda_2| = \mu_1 + |\mu_2| > -\mu_0 + n = k \). Again by Corollary 5.19, this proves Nagai’s conjecture fails. \( \square \)

6. **Nagai’s Conjecture for the Known Examples of Hyper-Kähler Manifolds**

At this point, we conclude with the proof of Nagai’s conjecture (Theorem 1.10) for the known cases of hyper-Kähler manifolds. In fact, as announced in the introduction, a stronger representation theoretic condition holds for all known cases. Specifically, the following holds (also stated as Theorem 1.14 in the introduction):

**Theorem 6.1.** Let \( X \) be a \( 2n \)-dimensional hyper-Kähler manifold of \( K3[n] \), Kum\( _n \), OG6, or OG10 type. Then any irreducible \( \mathfrak{g} \)-module component \( V_\mu \) occurring in the LLV decomposition of \( H^*(X) \) satisfies

\[
\mu_0 + \cdots + \mu_{r-1} + |\mu_r| \leq n.
\]
Remark 6.3. There are at least two other equivalent ways to state the condition (6.2). The first one is in terms of weight polytopes. The condition (6.2) is equivalent to the weight polytope of $V_\mu$ being contained in that of the Verbitsky component $V_{(n)}$:

$$\text{WP}(V_\mu) \subset \text{WP}(V_{(n)}).$$

This can be easily seen as follows. The weight polytope of the Verbitsky component $V_{(n)}$ has vertices $\pm n\varepsilon_0, \cdots, \pm n\varepsilon_r$, obtained by applying the Weyl group actions to the highest weight $n\varepsilon_0$. From it, one shows its weight polytope is

$$\text{WP}(V_{(n)}) = \{ \theta = \theta_0\varepsilon_0 + \cdots + \theta_r\varepsilon_r \in h^*_\mathfrak{g} : |\theta_0| + \cdots + |\theta_r| \leq n \}.$$  

In our case, $\mu$ is a dominant integral weight so we have an assumption $\mu_0 \geq \cdots \geq \mu_r \geq 0$. Hence (6.2) is equivalent to $\mu \in \text{WP}(V_{(n)})$, which is again equivalent to our condition on the weight polytopes. In this sense, the condition (6.2) in some sense means that the Verbitsky component is the dominant component among the LLV components arising in $H^*(X)$.

The second equivalent way to state the condition (6.2) is to use the notion of a cocharacter. Assume $\dim V = 2r + 3$ is odd. Let us denote $\varpi_r$ the fundamental weight associated to the spin representation $V_{\varpi_r}$. Then one can consider the cocharacter $\varpi_r^\vee := \frac{2(\varpi_r, \varpi_r)}{(\varpi_r, \varpi_r)}$ associated to it. Then the inequality (6.2) is equivalent to the inequality

$$\langle \mu, \varpi_r^\vee \rangle \leq (n\varepsilon_0, \varpi_r^\vee).$$

That is, the highest weight $\mu$ is again dominated in terms of the pairing with the cocharacter $\varpi_r^\vee$ associated to the spin representation. If $\dim V = 2r + 2$ is even, then we have to take care of the case $\mu_r < 0$, so (6.2) is in fact equivalent to two inequalities $\mu_0 + \cdots + \mu_{r-1} - \mu_r \leq n$ and $\mu_0 + \cdots + \mu_{r-1} + \mu_r \leq n$. This case, we have two half-spin representations associated to the fundamental weights $\varpi_{r-1} = \frac{1}{2}(\varepsilon_0 + \cdots + \varepsilon_{r-1} - \varepsilon_r)$ and $\varpi_r = \frac{1}{2}(\varepsilon_0 + \cdots + \varepsilon_r)$. Hence the condition (6.2) is equivalent to

$$\langle \mu, \varpi_{r-1}^\vee \rangle \leq (n\varepsilon_0, \varpi_{r-1}^\vee) \quad \text{and} \quad \langle \mu, \varpi_r^\vee \rangle \leq (n\varepsilon_0, \varpi_r^\vee).$$

This can be again interpreted as the highest weight $\mu$ is dominated in terms of the pairing with the cocharacters associated to the two half-spin representations.

Proof of Theorem 6.1. The inequality for OG6 and OG10 follows directly from the irreducible LLV decomposition in Theorem 1.1 items (3) and (4) respectively. Assume now that $X$ is a hyper-Kähler manifold of $K3^{[n]}$ type. We will in fact prove that every weight $\mu$ associated to $H^*(X)$ satisfies the desired inequality above. In this situation, we have the following generating series for the formal character (cf. Theorem 1.1(1)):

$$\sum_{n=0}^{\infty} \text{ch}(H^*(K3^{[n]}), \mathbb{R})) q^n = \prod_{m=1}^{11} \prod_{i=0}^{11} \frac{1}{(1 - x_i q^m)(1 - x_i^{-1} q^m)}.$$  

By definition, the coefficient of $q^n$ gives all the $q$-weights of $H^*(X)$. The weight $\mu = \mu_0\varepsilon_0 + \cdots + \mu_{11}\varepsilon_{11}$ corresponds to the monomial $x_0^{\mu_0} x_1^{\mu_1} \cdots x_{11}^{\mu_{11}}$ in the representation ring. Hence, the desired inequality $\mu_0 + \cdots + \mu_{11} \leq n$ is equivalent to saying that the $x_i$-degree of the coefficient of $q^n$ is $\leq n$. This is obvious from the form of the right hand side; whenever we increase the degree of $q$ by $m > 0$, then the degree of $x_i$ increases at most by $1(\leq m)$. Thus, for every monomial in the generating series, the $x_i$-degree is at most the $q$-degree. The claim follows.

For a $\text{Kum}_n$ type hyper-Kähler manifold $X$, the same argument applies. In this case, the generating series of the formal character of $H^*(X)$ is (cf. Theorem 1.1(2)):

$$\sum_{n=0}^{\infty} \text{ch}(H^*(\text{Kum}_n), \mathbb{R})) q^n = \sum_{d=1}^{\infty} J_d(d) \frac{B(q^d) - 1}{b_1 q},$$

where $B(q)$ is defined by

$$B(q) = \prod_{m=1}^{3} \left[ \prod_{l=0}^{3} \frac{1}{(1 - x_l q^m)(1 - x_l^{-1} q^m)} \right] \left[ (1 + x_0^{j_0} x_1^{j_1} x_2^{j_2} x_3^{j_3} q^m) \right].$$
Note that in the denominator we have
\[ b_1 = x_0 + \cdots + x_3 + x_0^{-1} + \cdots + x_3^{-1} + \sqrt{x_0 x_1 x_2 x_3} + \cdots + \sqrt{x_0 x_1 x_2 x_3}^{-1}. \]
Assume on the contrary that there exists some monomial \( x_0^{n_0} \cdots x_3^{n_3} q^n \) in the generating series such that \( \mu_0 + \cdots + \mu_3 \geq n + \frac{1}{2} \). After multiplying \( b_1 q \), it follows that some \( B(q^2) \) contains a monomial \( x_0^{\mu_0 + \frac{3}{2}} \cdots x_3^{\mu_3 + \frac{3}{2}} q^{n+1} \). This means that \( B(q) \) contains a monomial with \( x_i \)-degree at least \( \frac{3}{2} \) bigger than the \( q \)-degree. One can see without difficulty this cannot happen in \( B(q) \) defined as above. \( \square \)

Combining Theorem 6.1 with the representation theoretic formulation of Nagai’s conjecture (Theorem 5.4), we conclude Nagai’s conjecture holds for all currently known examples of hyper-Kähler manifolds.

**Corollary 6.4.** Nagai’s conjecture (1.9) holds for all one-parameter degenerations of compact hyper-Kähler manifolds of K3\([n]\), Kum\(_n\), OG6, or OG10 type. \( \square \)

It is natural to speculate that Nagai’s conjecture (or even the stronger inequality (6.2)) holds for any hyper-Kähler manifold. We do not have much to say in this direction. However, for completeness, we prove Nagai’s conjecture holds in general for low (\( \leq 8 \)) dimensional cases and for the case when \( b_2 = 3 \) (N.B. no such example is known to exist as of now).

**Proposition 6.5.** Nagai’s conjecture (1.9) holds when \( \dim X \leq 8 \).

**Proof.** The case \( b_2(X) = 3 \) is handled by Proposition 6.7 below. Thus, we can assume \( b_2(X) \geq 4 \), which allows us to use the criterion given by Theorem 5.4. From Proposition 2.34, every highest weight \( \mu \) in the LLV decomposition of the even cohomology \( H^{even}_X = \bigoplus_\nu V_{\mu}^{\nu} \) satisfies either \( \mu_0 + \mu_1 \leq n-1 \) or \( \mu = (n) \). The case \( \mu = (n) \) clearly satisfies \( \mu_0 + \mu_1 + |\mu_2| \leq n \). If \( n \leq 4 \), we get \( \mu_0 + \mu_1 \leq 3 \) and hence \(|\mu_2| \leq 1 \). This proves \( \mu_0 + \mu_1 + |\mu_2| \leq n \). \( \square \)

The case \( b_2(X) = 3 \) requires special attention (e.g. the signature on \( \bar{V} \) is definite, vs. indefinite in all the other cases). This case complements Theorem 5.4 which requires \( b_2(X) \geq 4 \). Assume
\[ b_2(X) = \dim \bar{V} = 3. \]
Then the rank of the simple Lie algebra \( \bar{g} = \mathfrak{so}(\bar{V}, q) (\approx \mathfrak{so}(3)) \) is \( r = 1 \), so irreducible \( \bar{g} \)-modules are classified by nonnegative half-integers \( \lambda \in \frac{1}{2} \mathbb{Z}_{\geq 0} \). The following lemma is an easy analogue of Lemma 5.11 (we omit the proof).

**Lemma 6.6.** Assume \( b_2(X) = 3 \). Let \( \lambda \in \mathbb{Z}_{\geq 0} \) be a dominant integral weight of \( \bar{g} \) and \( \rho_\lambda : \bar{g} \to \text{End}(\bar{V}_\lambda) \) the highest \( \bar{g} \)-module associated to it.

(i) If \( \nu_2 = 1 \), then \( \rho_\lambda(N) \) has nilpotency index \( \lambda \).

(ii) If \( \nu_2 = 2 \), then \( \rho_\lambda(N) \) has nilpotency index \( 2\lambda \). \( \square \)

**Proposition 6.7.** Nagai’s conjecture (1.9) holds when \( b_2(X) = 3 \).

**Proof.** From Lemma 6.6, the corresponding statement for Proposition 5.18 in this case is
\[ \nu_{2k} = \max\{ \lambda \nu_2 : \bar{V}_\lambda \subset H^{2k}(X) \}. \]
By Proposition 5.8, we know that if \( \nu_2 = 2 \) then \( \nu_{2k} = 2k \). Hence when \( \nu_2 = 1 \), we have \( \nu_{2k} = k \). \( \square \)

**Appendix A. Representation Theory of Simple Lie Algebras of Type BD**

We present a short review and fix notation for finite dimensional representation theory of simple Lie algebras of type BD. Throughout this section, we fix the notation \( \mathfrak{g} = \mathfrak{so}(V, q) \) for a special orthogonal Lie algebra associated to an arbitrary nondegenerate quadratic space \((V, q)\) over \( \mathbb{Q} \). Over the complex numbers \( \mathbb{C} \), there is only one quadratic space of dimension \( n \) up to isomorphism, so every type BD simple Lie algebra over \( \mathbb{C} \) is isomorphic to \( \mathfrak{so}(n, \mathbb{C}) \). Over the real numbers \( \mathbb{R} \), by Sylvester’s classification, quadratic forms on \( \mathbb{R}^n \) is classified by its signature \((a, b)\), so every type BD simple Lie algebra over \( \mathbb{R} \) is isomorphic to \( \mathfrak{so}(a, b) \).
Over the rational numbers \( \mathbb{Q} \), the case of interest here, the classification of quadratic forms on \( \mathbb{Q}^n \) is well understood, but more subtle (e.g., [O'M63]).

The LLV algebra of a hyper-Kähler manifold is of the form \( g = \mathfrak{so}(V,q) \) (see Theorem 2.7) for a rational quadratic space \( (V,q) \). We will review some representation theory facts in this simplest case of type BD Lie algebra. We will do the representation theory over \( \mathbb{Q} \) as much as possible. By definition, a finite dimensional \( \mathbb{Q} \)-vector space \( W \) is called a \( g \)-module, or a \( g \)-representation, if it is equipped with a Lie algebra homomorphism \( g \to \mathfrak{g}(W) \). Our main references for this appendix are [FH91] for representation theory over \( \mathbb{C} \), and Milne [Mil17] for that over \( \mathbb{Q} \).

A.1. **Type B.** Assume \( (V,q) \) is a rational quadratic space of odd dimension \( 2r + 1 \geq 3 \).

Fix any Cartan subalgebra \( h \subset g_\mathbb{C} \). Let \( 0, \pm \varepsilon_1, \cdots, \pm \varepsilon_r \) be the associated weights of the standard representation \( V_\mathbb{C} \) with respect to \( h \). We can choose a positive Weyl chamber appropriately so that it is generated by the fundamental weights

\[
\varpi_i = \varepsilon_1 + \cdots + \varepsilon_i \quad \text{for} \quad 1 \leq i \leq r - 1, \quad \varpi_r = \frac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_r).
\]

(A.1)

Let \( \Lambda \subset h_\mathbb{R}^\vee \) be the weight lattice of \( g \). It is a rank \( r \) lattice in the Euclidean space \( h_\mathbb{R}^\vee \) generated by the above fundamental weights. The intersection of the positive Weyl chamber and \( \Lambda \) is the monoid of dominant integral weights

\[
\Lambda^+ = \{ \lambda = \sum_{i=1}^{r} \lambda_i \varepsilon_i : \lambda_1 \geq \cdots \geq \lambda_r \geq 0, \quad \lambda_i - \lambda_j \in \mathbb{Z} \}.
\]

(A.2)

We will often denote a dominant integral weight \( \lambda = \sum_{i=1}^{r} \lambda_i \varepsilon_i \) simply as \( \lambda = (\lambda_1, \cdots, \lambda_r) \), and omit the zeros in the end for simplicity. Whenever we use this notation, we assume \( \lambda \) is a dominant integral weight and the conditions on \( \lambda_i \) above (A.2) are tacitly assumed.

Let \( \lambda \) be a dominant integral weight of \( g \). Over \( \mathbb{C} \), we always have a unique irreducible \( g_\mathbb{C} \)-module \( V_\lambda, \mathbb{C} \) with highest weight \( \lambda \). We call this a highest \( g_\mathbb{C} \)-module \( V_\lambda, \mathbb{C} \). Over \( \mathbb{Q} \), this is not always possible. However, in our case of \( g = \mathfrak{so}(V,q) \), we have a strong condition that the standard \( g_\mathbb{Q} \)-module \( V \) is defined over \( \mathbb{Q} \). This implies that the modules relevant to us are in fact defined over \( \mathbb{Q} \).

**Proposition A.3.** Let \( \lambda = (\lambda_1, \cdots, \lambda_r) \) be a dominant integral weight of \( g \). If \( \lambda_i \) are integers, then there exists a unique irreducible \( g \)-module \( V_\lambda \) such that its complexification \( (V_\lambda)_\mathbb{C} \) is isomorphic to the highest \( g_\mathbb{C} \)-module \( V_\lambda, \mathbb{C} \). We call this \( V_\lambda \) the highest \( g \)-module of weight \( \lambda \).

**Proof.** The orthogonal Schur-Weyl construction [FH91, Thm 19.22] realizes the highest \( g_\mathbb{C} \)-module \( V_\lambda, \mathbb{C} \) as an explicit tensor construction starting from the standard module \( V_\mathbb{C} \). This construction works over \( \mathbb{Q} \), and hence one can apply it to the rational \( g \)-module \( V \) and end up with a rational \( g \)-module \( V_\lambda \). This proves \( V_\lambda, \mathbb{C} \) is in fact defined over \( \mathbb{Q} \). Uniqueness is a general fact in representation theory over an arbitrary field (see, e.g., [Mil17, Thm 25.34]). \( \square \)

The highest \( g \)-modules associated to the fundamental weights (A.1) are most easily described. For \( 1 \leq i \leq r - 1 \), the highest \( g \)-module of weight \( \varpi_i \) is isomorphic to \( \Lambda^i V \), the \( i \)-th wedge power of the standard module \( V \). The highest module associated to the weight \( \varpi_r \) is exceptional; it is the spin \( g \)-module \( V_{\varpi_r, \mathbb{C}} \). Note that Proposition A.3 does not guarantee \( V_{\varpi_r, \mathbb{C}} \) is defined over \( \mathbb{Q} \). Indeed, it is completely possible that the spin module is not even defined over \( \mathbb{R} \) (see, e.g., [Del99]). However, one should be aware that \( V_{2\varpi_r} \) is defined over \( \mathbb{Q} \) and isomorphic to \( \Lambda^r V \). It is an irreducible \( g \)-module.

Any \( g \)-module \( W \) over \( \mathbb{Q} \) admits an associated \( \text{Spin}(V,q) \)-module structure and vice versa [Mil17, Thm 22.53]. The existence of the degree 2 isometry \( \text{Spin}(V,q) \to SO(V,q) \) says there are exactly half the irreducible \( SO(V,q) \)-modules than that of \( \text{Spin}(V,q) \)-modules. More specifically, \( V_\lambda \) admits an associated \( SO(V,q) \)-module structure if and only if \( \lambda \) is contained in the following submonoid of \( \Lambda^+ : \)

\[
\Lambda^+_\text{SO} = \{ \lambda = \sum_{i=1}^{r} \lambda_i \varepsilon_i : \lambda_1 \geq \cdots \geq \lambda_r \geq 0, \quad \lambda_i \in \mathbb{Z} \}.
\]

Note that this consists of precisely the dominant integral weights stated in Proposition A.3. Hence, every \( SO(V,q) \)-module is defined over \( \mathbb{Q} \).

The Weyl group \( \mathfrak{W} \) of \( g \) is the symmetry group of its root system consisting of permutations of the weights \( \varepsilon_1, \cdots, \varepsilon_r \) and their sign changes. More specifically, \( \mathfrak{W} \cong \mathfrak{S}_r \ltimes (\mathbb{Z}/2) \times r \) where \( \mathfrak{S}_r \) is the symmetric group of
order \( r \) and the the semidirect product is defined in terms of the group \( \mathcal{S}_r \) acting on \((\mathbb{Z}/2)^r\) by permuting factors. For every highest weight \( \lambda \), the set of weights of \( V_{\lambda,\mathbb{C}} \) is \( \mathfrak{W} \)-invariant as a subset in the Euclidean space \( \mathfrak{h}_\mathbb{C}^\vee \). Moreover, if we consider the convex hull in \( \mathfrak{h}_\mathbb{C}^\vee \) generated by all of the weights of \( V_{\lambda,\mathbb{C}} \), then we have a weight polytope \( WP(V_\lambda) \). The vertices of this polytope are exactly the points \( w, \lambda \) where \( w \in \mathfrak{W} \) varies through all the Weyl group actions. Some of them can coincide.

The Weyl dimension formula provides a convenient way to compute the dimension of the highest weight modules \( V_{\lambda,\mathbb{C}} \). The formula is as follows.

\[
\dim V_{\lambda,\mathbb{C}} = \prod_{1 \leq i < j \leq r} \frac{(\lambda + \rho, \epsilon_i + \epsilon_j) \cdot (\lambda + \rho, \epsilon_i - \epsilon_j)}{(\rho, \epsilon_i + \epsilon_j) \cdot (\rho, \epsilon_i - \epsilon_j)} \cdot \prod_{i=1}^r \frac{(\rho + \lambda, \epsilon_i)}{(\rho, \epsilon_i)}. \tag{A.4}
\]

Here \( \rho = \sum_{i=1}^r (r - i + \frac{1}{2}) \epsilon_i \) is half the sum of the positive roots and \((,\) is the standard Euclidean inner product on \( \mathfrak{h}_\mathbb{C}^\vee \) with respect to the basis \( \epsilon_i \), the Killing form. Of course, if \( V_{\lambda,\mathbb{C}} \) is defined over \( \mathbb{Q} \) with its \( \mathbb{Q} \)-dimension \( V_\lambda \) can be computed by exactly the same formula.

**A.2. Type D.** Assume \((V, q)\) is a rational quadratic space of even dimension \( 2r \geq 4 \). There is an analogue but slightly different story in this case. Again, start with fixing any Cartan subalgebra \( \mathfrak{h} \subset \mathfrak{g}_\mathbb{C} \). We have \( \pm \epsilon_1, \ldots, \pm \epsilon_r \) as the weights associated to the standard \( \mathfrak{g}_\mathbb{C} \)-module \( V_\mathbb{C} \) with respect to \( \mathfrak{h} \). Taking an appropriate positive Weyl chamber, we can choose the fundamental weights by

\[
\varpi_i = \epsilon_1 + \cdots + \epsilon_i \quad \text{for} \quad 1 \leq i \leq r - 2, \quad \varpi_{r-1} = \frac{1}{2}(\epsilon_1 + \cdots + \epsilon_{r-1} - \epsilon_r), \quad \varpi_r = \frac{1}{2}(\epsilon_1 + \cdots + \epsilon_r). \tag{A.5}
\]

The monoid of dominant integral weights is

\[
\Lambda^+ = \{ \lambda = \sum_{i=1}^r \lambda_i \epsilon_i : \lambda_1 \geq \cdots \geq \lambda_r \geq 0, \quad \lambda_i \in \frac{1}{2} \mathbb{Z}, \quad \lambda_i - \lambda_j \in \mathbb{Z} \}. \tag{A.6}
\]

We often denote \( \lambda = (\lambda_1, \ldots, \lambda_r) \) for a dominant integral weight \( \lambda = \sum_{i=1}^r \lambda_i \epsilon_i \), satisfying the condition (A.6).

We denote \( V_{\lambda,\mathbb{C}} \) the highest \( \mathfrak{g}_\mathbb{C} \)-module of weight \( \lambda \). A similar proposition on their field of definition holds for \( V_{\lambda,\mathbb{C}} \), but this time a bit more complicated than the previous one.

**Proposition A.7.** Let \( \lambda = (\lambda_1, \ldots, \lambda_r) \) be a dominant integral weight of \( \mathfrak{g} \). If \( \lambda_r = 0 \) then there exists a unique irreducible \( \mathfrak{g} \)-module \( V_\lambda \) such that its base change over \( \mathbb{C} \) is the highest module \( V_{\lambda,\mathbb{C}} \).

**Proof.** The proof is the same as above. Notice the conditions on \( \lambda \) are different for type B and type D. \( \square \)

Note that the orthogonal Schur–Weyl construction argument [FH91, Thm 19.22] still says \( V_{\lambda,\mathbb{C}} \oplus V_{\lambda',\mathbb{C}} \) is defined over \( \mathbb{Q} \) for \( \lambda, \lambda' \in \mathbb{Z} \setminus \{0\} \), where \( \lambda' = (\lambda_1, \ldots, \lambda_{r-1}, -\lambda_r) \). Hence one cannot say the highest \( \mathfrak{g}_\mathbb{C} \)-module \( V_{\lambda,\mathbb{C}} \) is always defined over \( \mathbb{Q} \) when \( \lambda_r \in \mathbb{Z} \setminus \{0\} \). However, only two cases can possibly arise:

i) There do exist irreducible \( \mathfrak{g} \)-modules \( V_\lambda \) and \( V_{\lambda'} \), whose complexification are the highest \( \mathfrak{g}_\mathbb{C} \)-modules \( V_{\lambda,\mathbb{C}} \) and \( V_{\lambda',\mathbb{C}} \).

ii) There does not exist any irreducible \( \mathfrak{g} \)-module whose complexification is the highest \( \mathfrak{g}_\mathbb{C} \)-module \( V_{\lambda,\mathbb{C}} \) (resp. \( V_{\lambda',\mathbb{C}} \)). Nonetheless, there exists a unique irreducible \( \mathfrak{g} \)-module \( V_\lambda = V_{\lambda'} \) whose complexification is \( V_{\lambda,\mathbb{C}} \oplus V_{\lambda',\mathbb{C}} \).

The highest \( \mathfrak{g} \)-modules associated to the fundamental weights (A.5) are \( V_{\varpi_i} = \wedge^i V \) for \( 1 \leq i \leq r - 2 \). For \( i = r - 1, r \), we get the two half-spin representations \( V_{\varpi_{r-1},\mathbb{C}} \) and \( V_{\varpi_r,\mathbb{C}} \). Again, it is totally possible these half-spin representations are not defined over \( \mathbb{Q} \) [Del99]. We also note the isomorphisms \( \wedge^{r-1} V = V_{\varpi_{r-1} + \varpi_r} \) and \( \wedge^r V = V_{2\varpi_{r-1}} \oplus V_{2\varpi_r} \). In particular, \( \wedge^{r-1} V \) is an irreducible \( \mathfrak{g} \)-module whereas \( \wedge^r V \) is not.

Any \( \mathfrak{g} \)-module \( W \) over \( \mathbb{Q} \) admits an associated \( \text{Spin}(V, q) \)-module structure and vice versa. There exists a degree 2 isogeny \( \text{Spin}(V, q) \rightarrow \text{SO}(V, q) \). This says \( V_\lambda \) admits an associated \( \text{SO}(V, q) \)-module structure if and only if \( \lambda \) is contained in

\[
A^+_{\text{SO}} = \{ \lambda = \sum_{i=1}^r \lambda_i \epsilon_i : \lambda_1 \geq \cdots \geq \lambda_r \geq 0, \quad \lambda_i \in \mathbb{Z} \}.
\]
In this case, the center of Spin(V, q) is isomorphic to (Z/2)^2 and hence there exists a further degree 2 isogeny SO(V, q) → PSO(V, q). Therefore, there are more possibility of Q-algebraic groups with the associated Lie algebra g.

The Weyl group W of g consists of permutations of the weights ε_1, · · · , ε_r and even number of their sign changes. The group W is an index 2 subgroup of Σ_r × (Z/2)^r, consisting of elements of even number of 1’s in (Z/2)^r. The set of weights of V_λ is W-invariant, and generates a convex hull WP(V_λ), the weight polytope of V_λ. The vertices of WP(V_λ) are exactly the points w.λ where w ∈ W varies through all the Weyl group actions.

For any dominant integral weight λ, the Weyl dimension formula for this case has the following form:

\[ \dim V_{λ,C} = \prod_{1 \leq i < j \leq r} \frac{(λ + ρ, ε_i + ε_j) \cdot (λ + ρ, ε_i - ε_j)}{(ρ, ε_i + ε_j) \cdot (ρ, ε_i - ε_j)}. \] (A.8)

Here \( ρ = \sum_{i=1}^{r} (r - i)ε_i \) denotes again half the sum of the positive roots and (,) is the standard Euclidean inner product on \( h^*_K \) with respect to the basis \( ε_i \). As an example, in this paper, we have used the Weyl dimension formula to generate Table 1 for hyper-Kähler manifolds of OG10 type. Since, in this case, the rank of g is r = 13, our computations were computer-aided. Finally, for the type D case, we provide the following lemma about the dimension comparison of \( V_λ \) (cf. [FH91, Ex. 24.9]). It is used in §3.4 for the study of the LLV decomposition of OG10 hyper-Kähler manifolds.

**Lemma A.9.** Let \( λ, μ \) be dominant integral weights of g. Then \( \dim V_{λ+μ,C} \geq \dim V_{λ,C} \).

**Appendix B. Representation ring and restriction representations**

Since many of our results involve several different Lie algebras and heavily depends on the relation between their representation theory, we provide a separate section to discuss this topic.

**B.1. Representation ring and restriction representations.** Let g be a reductive Lie algebra over \( \mathbb{Q} \). Recall that a (rational) g-module is a finite dimensional rational vector space \( V \) equipped with a Lie algebra homomorphism \( g \rightarrow gl(V) \). We define a complex g-module by a finite dimensional complex vector space V equipped with a Lie algebra homomorphism \( g_C \rightarrow gl(V) \). Notice that the notion of a complex g-module is nothing but just a \( g_C \)-module. If we have a rational g-module \( V_Q \), then its complexification \( (V_Q)_C \) is clearly a complex g-module. On the other hand, not every complex g-module can be obtained by the complexification of a rational g-module.

Let \( \text{Rep}(g) \) and \( \text{Rep}_C(g) \) ( \( = \text{Rep}(g_C) \)) be the categories of finite dimensional rational g-modules and complex g-modules, respectively. Since we have assumed g is reductive, both categories are semisimple, i.e., every object in the category is completely reducible. The discussion in the previous paragraph implies there exists a complexification functor

\[ \text{Rep}(g) \rightarrow \text{Rep}_C(g). \]

Consider the Grothendieck ring \( K(g) \) and \( K_C(g) \) of the categories \( \text{Rep}(g) \) and \( \text{Rep}_C(g) \), respectively. These rings are called the representation ring (resp. complex representation ring) of g. Since \( \text{Rep}(g) \) (resp. \( \text{Rep}_C(g) \)) is semisimple, the representation ring \( K(g) \) (resp. \( K_C(g) \)) coincides with the abelianization of the ring of isomorphism classes of g-modules (resp. complex g-modules). Moreover, the above complexification functor induces an injective ring homomorphism (see [Mil17, §25.d])

\[ K(g) \hookrightarrow K_C(g). \] (B.1)

Thus, to describe the structure of (rational or complex) g-modules up to isomorphism, it is enough to describe them as elements in \( K_C(g) \).

**Proposition B.2.** Let V be a g-module. Then the g-module structure of V is completely determined by an element \([V_C] \in K_C(g)\) in the complex representation ring.

The structure of the representation ring \( K_C(g) \) for simple Lie algebras g is completely understood. It is related to the character theory and weights of g-modules. Fix a Cartan subalgebra of g and let \( Λ \) be the weight lattice of g. Consider its group ring \( Z[Λ] \). To use a multiplicative notation for the multiplication operation in \( Z[Λ] \), we use a notation \( e^μ \in Z[Λ] \) to represent \( μ \in Λ \) as an element in \( Z[Λ] \).
The weight lattice $\Lambda$ of it is generated by the fundamental weights $\varpi$. Theorem B.4 gives us the isomorphism $K \cong Z[\Lambda]$ as follows:

B.1.2. Restriction representations. A direct but interesting consequence of the above discussions is the following.

Definition B.3. Let $V$ be any complex $g$-module. Consider its weight decomposition $V = \bigoplus \mu V(\mu)$, where $V(\mu)$ indicates the weight $\mu$ subvector space of $V$. We define the formal character map of $g$ by a ring homomorphism

$$\text{ch} : K_g(g) \rightarrow Z[\Lambda], \quad [V] \mapsto \sum_{\mu} \dim V(\mu) e^\mu.$$

We recall the following well known result (e.g. [FH91, §23]).

Theorem B.4. The formal character map $\text{ch}$ is injective, and the image of it is the Weyl group invariant ring $Z[\Lambda]_W$. That is, $\text{ch} : K_g(g) \rightarrow Z[\Lambda]_W$ is a ring isomorphism.

B.1.1. Representation ring of type BD simple Lie algebras. Now let us specialize our discussion to type BD simple Lie algebras. That is, we assume $g = so(V,q)$ for a rational quadratic space $(V,q)$.

Assume $\dim V = 2r + 1$ is odd for $r \geq 1$ (Case B$_r$). The complexification of $g$ is $g_C = so(2r+1,\mathbb{C})$. Recall from Section A that the weight lattice $\Lambda$ of it is generated by the fundamental weights

$$\varpi_1 = \varepsilon_1, \quad \varpi_2 = \varepsilon_1 + \varepsilon_2, \quad \ldots, \quad \varpi_{r-1} = \varepsilon_1 + \cdots + \varepsilon_{r-1}, \quad \varpi_r = \frac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_r).$$

Let us simply write $x_i = e^{\varepsilon_i}$ for $i = 1, \ldots, r$. Then we can describe the group ring $Z[\Lambda]$ explicitly as

$$Z[\Lambda] = Z[x_1^{\pm 1}, \ldots, x_r^{\pm 1}, (x_1 \cdots x_r)^{\pm \frac{1}{2}}]. \quad (B.5)$$

Recall the Weyl group of $g$ is isomorphic to $W_{2r+1} \cong \mathfrak{S}_r \times (\mathbb{Z}/2)^{r-1}$. It acts on $Z[x_1^{\pm 1}, \ldots, x_r^{\pm 1}, (x_1 \cdots x_r)^{\pm \frac{1}{2}}]$ as follows: $\sigma \in \mathfrak{S}_r$ acts as a permutation on $x_1, \ldots, x_r$, and $1 \in \mathbb{Z}/2$ in the $i$-th factor $\mathbb{Z}/2$ acts as $x_i \mapsto x_i^{-1}$.

Finally, Theorem B.4 completes the explicit description of $K_g(g)$ by

$$\text{ch} : K_g(g) \cong Z[x_1^{\pm 1}, \ldots, x_r^{\pm 1}, (x_1 \cdots x_r)^{\pm \frac{1}{2}}]^{W_{2r+1}}.$$

Now assume $\dim V = 2r$ is even for $r \geq 2$ (Case D$_r$). The complexification of $g$ is $g_C = so(2r,\mathbb{C})$. The weight lattice $\Lambda$ of it is generated by the fundamental weights

$$\varpi_1 = \varepsilon_1, \quad \cdots, \quad \varpi_{r-2} = \varepsilon_1 + \cdots + \varepsilon_{r-2}, \quad \varpi_{r-1} = \frac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_{r-1} - \varepsilon_r), \quad \varpi_r = \frac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_r).$$

Let us also write $x_i = e^{\varepsilon_i}$ for $i = 1, \ldots, r$. Then the group ring $Z[\Lambda]$ becomes the same as above:

$$Z[\Lambda] = Z[x_1^{\pm 1}, \ldots, x_r^{\pm 1}, (x_1 \cdots x_r)^{\pm \frac{1}{2}}].$$

However, the Weyl group becomes smaller. The Weyl group $W_{2r}$ in this case is an order 2 subgroup of $\mathfrak{S}_r \times (\mathbb{Z}/2)^{r-1}$, consisting of elements of even number of 1’s in $(\mathbb{Z}/2)^{r-1}$. It acts on $Z[x_1^{\pm 1}, \ldots, x_r^{\pm 1}, (x_1 \cdots x_r)^{\pm \frac{1}{2}}]$ in the same way as above. Therefore, Theorem B.4 gives us the isomorphism

$$\text{ch} : K_g(g) \cong Z[x_1^{\pm 1}, \ldots, x_r^{\pm 1}, (x_1 \cdots x_r)^{\pm \frac{1}{2}}]^{W_{2r}}.$$

B.1.2. Restriction representations. A direct but interesting consequence of the above discussions is the following.

Proposition B.6. Let $(V,q)$ be a rational quadratic space and $T \subset V$ a subvector space with $\dim V = 2r + 1$ and $\dim T = 2r$. Set $g = so(V,q)$ and $m = so(T,q)$. Then the restriction representation functor $\text{Res} : \text{Rep}(g) \rightarrow \text{Rep}(m)$ induces an injective ring homomorphism on the level of representation rings. That is, the following diagram commutes with all the horizontal arrows injective.

$$\begin{array}{ccc}
K(g) & \xrightarrow{\text{Res}} & K(m) \\
\downarrow & & \downarrow \\
K_C(g) & \xrightarrow{\text{Res}} & K_C(m) \\
\text{ch} & \cong & \text{ch} \cong \\
Z[x_1^{\pm 1}, \ldots, x_r^{\pm 1}, (x_1 \cdots x_r)^{\pm \frac{1}{2}}]^{W_{2r+1}} & \cong & Z[x_1^{\pm 1}, \ldots, x_r^{\pm 1}, (x_1 \cdots x_r)^{\pm \frac{1}{2}}]^{W_{2r}}.
\end{array}$$
Some special branching rules for $B.2.2.$

\[ \lambda \\text{ so} \lambda \\text{ and} \\lambda \] and $B.2.1.$ often easier to deal with. We collect a few branching rules for type BD Lie algebras, which are useful for us.

above the theoretical framework of restriction representations, more explicit combinatorial descriptions are describing how the restriction representation of the two Lie algebras $U$ of the Mukai completion, i.e. the cohomology of compact hyper-Kähler manifold. Classically, this component is viewed as a “symmetric part” of the second cohomology as its $2k$-th degree part is isomorphic to the $k$-th symmetric power of the second degree part. We can recover this fact in the following way. Assume that we are in the standard set-up of the Mukai completion, i.e.

\[ (V, q) = (\bar{V}, \bar{q}) \oplus U, \]

where $U$ is the $2$-dimensional hyperbolic quadratic space, and denote $\bar{g} = so(\bar{V}, \bar{q}).$ Then we have an equality $V = \bar{V} \oplus \mathbb{Q}^2$ (since we are interested in the $so(\bar{V}, \bar{q})$-structure, the precise structure on the second component $\mathbb{Q}^2$ does not matter). One can compute

\[ \text{Sym}^k V = \text{Sym}^k (\bar{V} \oplus \mathbb{Q}^2) = \text{Sym}^k \bar{V} \oplus 2 \text{Sym}^{k-1} \bar{V} \oplus 3 \text{Sym}^{k-2} \bar{V} \oplus \cdots \oplus k \bar{V} \oplus (k + 1)\mathbb{Q} \]

That is, in the set-up of the proposition, if $W$ is a $g$-module then the $m$-module structure on $W$ by restriction representation determines its $g$-module structure. In particular, since $b_2$ is even for $K3$ surfaces, notice that this applies to case of the LLV algebra $m = so(4, 20)$ for $K3$ surfaces and the LLV algebra $g = so(4, 21)$ for $K3^{[g]}$ (and similarly, for the Kum_{a} series). This fact plays a key role in Section 3.

B.2. Some explicit examples of branching rules. A branching rule is simply a combinatorial rule describing how the restriction representation of the two Lie algebras $m \subset g$ behave. Although we discussed above the theoretical framework of restriction representations, more explicit combinatorial descriptions are often easier to deal with. We collect a few branching rules for type BD Lie algebras, which are useful for us.

B.2.1. The branching rule of $so(n, \mathbb{C}) \subset so(n + 1, \mathbb{C}).$ Let us consider the branching rule of $so(n, \mathbb{C}) \subset so(n + 1, \mathbb{C}).$ We temporarily assume everything is over $\mathbb{C}$ for this discussion. However, applying (B.1), one also concludes exactly the same branching rule for rational Lie algebras. Denote $V = \mathbb{C}^{n+1}$ and $W = \mathbb{C}^n$ for the standard representations of $so(n + 1)$ and $so(n).$ In [FH91, §25.3], there is an explicit branching rule describing how the highest $so(n+1)$-module $V_{\lambda}$ of weight $\lambda$ splits as a direct sum of irreducible $so(n)$-modules. The description is as follows.

Assume $n = 2r$ and let $\lambda = (\lambda_1, \cdots, \lambda_r)$ be a dominant integral weight of $so(2r + 1).$ Then we have an $so(2r)$-module irreducible decomposition of the highest $so(2r + 1)$-module

\[ V_{\lambda} = \bigoplus_{\lambda'} W_{\lambda'}, \]

where $\lambda' = (\lambda_1', \cdots, \lambda_r')$ runs through all the $so(2r)$-dominant integral weights with

\[ \lambda_1 \geq \lambda_1' \geq \lambda_2 \geq \lambda_2' \cdots \geq \lambda_{r-1} \geq \lambda_{r-1}' \geq \lambda_r \geq |\lambda_r'| \geq 0, \]

and $\lambda_1$ and $\lambda_1'$ are simultaneously all integers or half-integers.

Assume $n = 2r - 1$ and let $\lambda = (\lambda_1, \cdots, \lambda_r)$ be a dominant integral weight of $so(2r).$ Then we have an $so(2r - 1)$-module irreducible decomposition of the highest $so(2r)$-module

\[ V_{\lambda} = \bigoplus_{\lambda'} W_{\lambda'}, \]

where $\lambda' = (\lambda_1', \cdots, \lambda_{r-1}')$ runs through all the $so(2r - 1)$-dominant integral weights with

\[ \lambda_1 \geq \lambda_1' \geq \lambda_2 \geq \lambda_2' \cdots \geq \lambda_{r-1} \geq |\lambda_r| \geq 0, \]

and $\lambda_1$ and $\lambda_1'$ are simultaneously all integers or half-integers.

B.2.2. Some special branching rules for $m \subset g.$ Let $(V, q)$ be a rational quadratic space and $W \subset V$ be a subvector space. Set $g = so(V, q)$ and $m = so(W, q)$ be rational Lie algebras. Since any $W \subset V$ has its orthogonal complement, evidently we have an inclusion $m \subset g.$ Applying the above discussion on the branching rule of $so(n, \mathbb{C}) \subset so(n+1, \mathbb{C})$ several times (with the aid of (B.1)), we can get an explicit branching rule for $m \subset g.$

However, in the two special cases $V(k)$ and $V_{1, \ldots, 1}$, there is another easier way to obtain a branching rule. Let us first consider the case of the $g$-module $V(k)$. This is precisely the case for the Verbitsky component in the cohomology of compact hyper-Kähler manifold. Classically, this component is viewed as a “symmetric power” of the second cohomology as its $2k$-th degree part is isomorphic to the $k$-th symmetric power of the second degree part. We can recover this fact in the following way. Assume that we are in the standard set-up of the Mukai completion, i.e.

\[ (V, q) = (\bar{V}, \bar{q}) \oplus U, \]

where $U$ is the $2$-dimensional hyperbolic quadratic space, and denote $\bar{g} = so(\bar{V}, \bar{q}).$ Then we have an equality $V = \bar{V} \oplus \mathbb{Q}^2$ (since we are interested in the $so(\bar{V}, \bar{q})$-structure, the precise structure on the second component $\mathbb{Q}^2$ does not matter). One can compute

\[ \text{Sym}^k V = \text{Sym}^k (\bar{V} \oplus \mathbb{Q}^2) = \text{Sym}^k \bar{V} \oplus 2 \text{Sym}^{k-1} \bar{V} \oplus 3 \text{Sym}^{k-2} \bar{V} \oplus \cdots \oplus k \bar{V} \oplus (k + 1)\mathbb{Q} \]
as a \( g \)-module decomposition. Now it is well known \( \text{Sym}^{k} V = \text{Sym}^{k-2} V \oplus V_{(k)} \) as \( g \)-modules, so this leads us to the identity
\[
V_{(k)} = \text{Sym}^{k} V \oplus 2\text{Sym}^{k-1} V \oplus 2\text{Sym}^{k-2} V \oplus \cdots \oplus 2V \oplus 2\mathbb{Q}.
\]
In particular, this recovers the symmetric power description of the Verbitsky component \( V_{(k)} \). If one also wants to capture the degree of the components, then one can consider the decomposition \( V = \mathbb{Q}(-1) \oplus V \oplus \mathbb{Q}(1) \) instead, where \( \mathbb{Q}(-1) \) and \( \mathbb{Q}(1) \) denote the \( \pm 2 \) eigenspaces of the “grading operator” \( h \) (see Section 2, esp. (2.1)).

The branching rule for the \( g \)-module \( V_{(1,\ldots,1)} = V_{(k)} \) ( \( k \) times of 1’s) will be used when we discuss the LLV decomposition of hyper-Kähler manifolds of OG6 type. Here we assume \( m = \mathfrak{so}(W,q) \) with \( \dim V - \dim W = m \). Thus, we can write \( V = W \oplus \mathbb{Q}^m \) and get
\[
V_{(1^k)} = \bigwedge^k V = \bigwedge^k(W \oplus \mathbb{Q}^m)
\]
\[
= \bigwedge^k W \oplus m \bigwedge^{k-1} W \oplus \binom{m}{2} \bigwedge^{k-2} W \oplus \cdots \oplus \binom{m}{k-1} W \oplus \binom{m}{k} \mathbb{Q}
\]
\[
= W_{(1^k)} \oplus mW_{(1^{k-1})} \oplus \cdots \oplus \binom{m}{k-1} W \oplus \binom{m}{k} \mathbb{Q}.
\]
This gives us the decomposition of \( V_{(1^k)} \) into a direct sum of irreducible \( m \)-modules.

References


