Problem 1. Show that if \( f(x) \) is a polynomial whose degree is less than \( n \), then the fraction \( \frac{f(x)}{(x-x_1)(x-x_2)\cdots(x-x_n)} \) where \( x_1, x_2, \ldots, x_n \) are \( n \) distinct numbers, can be represented as a sum of \( n \) partial fractions \( \frac{A_1}{x-x_1} + \cdots + \frac{A_n}{x-x_n} \) where \( A_1, \ldots, A_n \) are constants.

As an application, let \( f(x) \) be a monic polynomial of degree \( n \) with distinct zeros \( x_1, x_2, \ldots, x_n \). Let \( g(x) \) be any monic polynomial of degree \( n-1 \). Show that \( \sum_{j=1}^{n} g(x_j) = 1 \).

Problem 2. If \( x_1, x_2, \ldots, x_n \) are distinct numbers, and \( y_1, y_2, \ldots, y_n \) are any numbers, prove that there is a unique polynomial \( P(x) \) of degree at most \( n-1 \) such that \( P(x_j) = y_j, j = 1, 2, \ldots, n \).

Problem 3. If \( n > 1 \), show that \( (x+1)^n - x^n - 1 = 0 \) has a multiple root if and only if \( n-1 \) is divisible by 6.

Problem 4. Find all polynomials whose coefficients are equal to either 1 or -1 and whose zeros are all real.

Problem 5. Prove that for every positive integer \( n \),
\[
\tan \frac{\pi}{2n+1} \tan \frac{2\pi}{2n+1} \cdots \tan \frac{n\pi}{2n+1} = \sqrt{2n+1}.
\]

Problem 6. Determine all polynomials \( P(x) \) with real coefficients satisfying \( (P(x))^n = P(x^n) \) for all \( x \in \mathbb{R} \), where \( n > 1 \) is a fixed integer.

Problem 7. Let \( P(z) \) and \( Q(z) \) be polynomials with complex coefficients of degree greater than or equal to 1 with the property that \( P(z) = 0 \) if and only if \( Q(z) = 0 \) and \( P(z) = 1 \) if and only if \( Q(z) = 1 \). Prove that the polynomials are equal.

Problem 8. Let \( P(x) \) be a polynomial of degree \( n \geq 3 \) whose zeros \( x_1 < x_2 < \cdots < x_n \) are real. Prove that
\[
P'(\frac{x_1 + x_2}{2}) P'(\frac{x_{n-1} + x_n}{2}) \neq 0.
\]

Problem 9. Let \( a_1, \ldots, a_n \) be positive real numbers. Prove that the polynomial \( P(x) = x^n - a_1 x^{n-1} - a_2 x^{n-2} - \cdots - a_n \) has a unique positive zero.

Problem 10. For a polynomial \( P(x) = (x-x_1)(x-x_2)\cdots(x-x_n) \), with distinct real zeros \( x_1 < x_2 < \cdots < x_n, n \geq 3 \), we set \( \delta(P(x)) = \min_i(x_{i+1} - x_i) \). Prove \( \delta(P'(x)) > \delta(P(x)) \).

Problem 11. Associate to a prime the polynomial whose coefficients are the decimal digits of the prime. Prove that this polynomial is always irreducible over \( \mathbb{Z}[x] \).

Problem 12. Let \( p < m \) be two positive integers. Prove that
\[
\det \begin{pmatrix}
\binom{m}{0} & \binom{m}{1} & \cdots & \binom{m}{p} \\
\binom{m+1}{0} & \binom{m+1}{1} & \cdots & \binom{m+1}{p} \\
\vdots & \vdots & \ddots & \vdots \\
\binom{m+p}{0} & \binom{m+p}{1} & \cdots & \binom{m+p}{p}
\end{pmatrix} = 1.
\]