**Problem 1.** Let \( a, b, c, d \) be real numbers such that \( c \neq 0 \) and \( ad - bc = 1 \). Prove that there exist \( u \) and \( v \) such that

\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & -u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} \begin{pmatrix} 1 & -v \\ 0 & 1 \end{pmatrix}.
\]

**Problem 2.** Calculate the \( n \)th power of the \( m \times m \) matrix

\[
\begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda \\ 0 & 0 & \cdots & 1 \end{pmatrix}.
\]

**Problem 3.** Derive the formula for the determinant of a circulant matrix

\[
\begin{pmatrix} x_1 & x_2 & x_3 & \cdots & x_n \\ x_n & x_1 & x_2 & \cdots & x_{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_3 & x_4 & x_5 & \cdots & x_2 \\ x_2 & x_3 & x_4 & \cdots & x_1 \end{pmatrix}
\]

\[
\det \begin{pmatrix} x_1 & x_2 & x_3 & \cdots & x_n \\ x_n & x_1 & x_2 & \cdots & x_{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_3 & x_4 & x_5 & \cdots & x_2 \\ x_2 & x_3 & x_4 & \cdots & x_1 \end{pmatrix} = (-1)^{n-1} \prod_{j=0}^{n-1} \left( \sum_{k=1}^{n} \zeta^{jk} x_k \right),
\]

where \( \zeta = e^{2\pi i/n} \).

**Problem 4.** Compute the determinant of the \( n \times n \) matrix \( A = (a_{ij})_{ij} \) where \( a_{ij} = (-1)^{|i-j|} \) if \( i \neq j \) and \( a_{ii} = 2 \).

**Problem 5.** Prove that for any integers \( x_1, x_2, ..., x_n \) and positive integers \( k_1, k_2, ..., k_n \), the determinant

\[
\det \begin{pmatrix} x_1^{k_1} & x_2^{k_1} & \cdots & x_n^{k_1} \\ x_1^{k_2} & x_2^{k_2} & \cdots & x_n^{k_2} \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{k_n} & x_2^{k_n} & \cdots & x_n^{k_n} \end{pmatrix}
\]

is divisible by \( n! \).

**Problem 6.** Let \( P(t) \) be a polynomial of even degree with real coefficients. Prove that \( f(X) = P(X) \) defined on the set of \( n \times n \) matrices is not onto.

**Problem 7.** Let \( A = (a_{ij})_{ij} \) be an \( n \times n \) such that \( \sum_{j=1}^{n} |a_{ij}| < 1 \) for each \( i \). Prove that \( I - A \) is invertible.

**Problem 8.** Let \( A \) be an \( n \times n \) matrix such that there exists a positive integer \( k \) for which \( kA^{k+1} = (k+1)A^k \). Prove that \( A - I \) is invertible and find its inverse.

**Problem 9.** A linear map \( A \) on the \( n \)-dimensional vector space \( V \) is called an involution if \( A^2 = I \).
a. Prove that for every involution $A$ on $V$ there exists a basis of $V$ consisting of eigenvectors of $A$.

b. Find the maximal number of distinct pairwise commuting involutions.

**Problem 10.** Find the $2 \times 2$ matrices with real entries that satisfy the equation

$$X^3 - 3X^2 = \begin{pmatrix} -2 & -2 \\ -2 & -2 \end{pmatrix}.$$

**Problem 11.** Let $x_1, x_2, \ldots, x_n$ be differentiable (real-valued) functions of a single variable $t$ that satisfy

$$\frac{dx_1}{dt} = a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n,$$

$$\frac{dx_2}{dt} = a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n,$$

$$\vdots$$

$$\frac{dx_n}{dt} = a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n,$$

for constants $a_{ij} > 0$. Suppose for all $i$ that $x_i(t) \to 0$ as $t \to \infty$. Are the functions $x_i$ necessarily linearly dependent?

**Problem 12.** Let $A$ be a $4 \times 4$ matrix such that each entry of $A$ is either 2 or $-1$. Let $d = \det(A)$. Show that $d$ is divisible by 27.

**Problem 13.** For any vector $v$ in $\mathbb{R}^n$ and permutation $\sigma$ of $\{1, 2, \ldots, n\}$, define $\sigma(v) = (x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)})$. What are the possibilities for the dimension of the space spanned by $\sigma(v)$ such that $\sigma$ is a permutation?

**Problem 14.** Let $f_1, f_2, \ldots, f_n$ be linearly independent, differentiable functions. Prove that some $n - 1$ of their derivatives $f_1', f_2', \ldots, f_n'$ are independent.