INDUCTION AND PIGEON HOLE; EXTREMAL CONFIGURATIONS

ROBERT HOUGH

Problem 1. Let $x_1, x_2, ..., x_n, y_1, y_2, ..., y_m$ be positive integers, $n, m > 1$. Assume that $x_1 + x_2 + \cdots + x_n = y_1 + y_2 + \cdots + y_m < mn$. Prove that in the equality one can suppress some but not all of the terms, with equality preserved.

Problem 2. Given a sequence of integers $x_1, x_2, ..., x_n$ whose sum is 1, prove that exactly one of the cyclic shifts

$$x_1, x_2, ..., x_n;\ x_2, ..., x_n, x_1;\ ...;\ x_n, x_1, ..., x_{n-1}$$

has all of its partial sums positive.

Problem 3. Show that every positive integer can be written as a sum of distinct terms of the Fibonacci sequence $F_0 = 0, F_1 = 1$, and $F_{n+1} = F_n + F_{n-1}$.

Problem 4. Prove $F_{2n+1} = F_{n+1}^2 + F_n^2$.

Problem 5. Prove $F_{3n} = F_{n+1}^3 + F_n^3 - F_{n-1}^3$.

Problem 6. Prove that any integer can be represented as $\pm 1^2 \pm 2^2 \pm \cdots \pm n^2$ for some positive integer n and some choice of signs.

Problem 7. Let $x_1, x_2, ...$ be a sequence of integers such that

$$1 = x_1 < x_2 < x_3 < \cdots, \qquad x_{n+1} \leqslant 2n.$$

Show that every positive integer k is equal to $x_i - x_j$ for some i and j.

Problem 8. Let $x_1, x_2, ..., x_k$ be real numbers such that the set $A = \{\cos(n\pi x_1) + \cdots + \cos(n\pi x_k) : n \geqslant 1\}$. Prove that all of the x_i are rational numbers.

Problem 9. Draw the diagonals of a 21-gon. Prove that at least one angle of less than 1 degree is formed.

Problem 10. Consider a planar region of area 1, obtained as the union of finitely many disks. Prove that from these disks we can select some that are mutually disjoint and have total area at least $\frac{1}{9}$.

Problem 11. Use induction to prove that $2!4! \cdots (2n)! \geqslant ((n+1)!)^n$.

Problem 12. The Euclidean plane is divided into regions by drawing a finite number of straight lines. Show that it is possible to color each of these regions either red or blue in such a way that no two adjacent regions have the same color.

Problem 13. Given an even number of points in the plane, is there a line in the plane such that half of the points are on either side?