## COMBINATORICS

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Problem 1. Let $a_{1}, a_{2}, \ldots, a_{n}$ be a permutation of $1,2, \ldots, n$. We call $a_{i}$ a large integer if $a_{i}>a_{j}$ for all $i<j<n$. Find the average number of large integers over all permutations of the first $n$ positive integers.

Problem 2. What is the largest number of internal right angles that an $n$-gon (convex or not, with non-self-intersecting boundary) can have.
Problem 3. Prove that

$$
\binom{2 k}{k}=\frac{2}{\pi} \int_{0}^{\frac{\pi}{2}}(2 \sin \theta)^{2 k} d \theta
$$

Problem 4. Consider the $n \times n$ matrix

$$
A=\left(\begin{array}{ccccc}
1 & 1 & 1 & \cdots & 1 \\
0 & 1 & 1 & \cdots & 1 \\
0 & 0 & 1 & \cdots & 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1
\end{array}\right) .
$$

Compute the matrix $A^{k}, k \geqslant 1$.
Problem 5. For an arithmetic progression $a_{1}, a_{2}, \ldots, a_{n}, \ldots$ let $S_{n}=a_{1}+a_{2}+\cdots+a_{n}$, $n \geqslant 1$. Prove that

$$
\sum_{k=0}^{n}\binom{n}{k} a_{k+1}=\frac{2^{n}}{n+1} S_{n+1}
$$

Problem 6. The $q$-binomial coefficient is defined by

$$
\binom{n}{m}_{q}=\frac{\left(q^{n}-1\right)\left(q^{n-1}-1\right) \ldots\left(q^{n-m+1}-1\right)}{\left(q^{m}-1\right)\left(q^{m-1}-1\right) \ldots(q-1)}
$$

Prove

$$
\sum_{k=0}^{n}(-1)^{k} q^{\frac{k(k-1)}{2}}\binom{n}{k}_{q}=0
$$

Problem 7. Prove Newton's binomial formula

$$
[x+y]_{n}=\sum_{k=0}^{n}\binom{n}{k}[x]_{k}[y]_{n-k}
$$

where $[x]_{n}=x(x-1) \cdots(x-n+1)$.
Problem 8. Prove the quantum binomial coefficients satisfy

$$
\binom{m+n}{k}_{q}=\sum_{j=0}^{k} q^{(m-j)(k-j)}\binom{m}{j}_{q}\binom{n}{k-j}_{q} .
$$

Problem 9. For a positive integer $n$, denote by $S(n)$ the number of choices of the signs + or - such that $\pm 1 \pm 2 \pm 3 \pm \ldots \pm n=0$. Prove that

$$
S(n)=\frac{2^{n-1}}{\pi} \int_{0}^{2 \pi} \cos t \cos 2 t \cdots \cos n t d t
$$

Problem 10. The distinct positive integers $a_{1}, a_{2}, \ldots, a_{n}, b_{1}, b_{2}, \ldots, b_{n}$ with $n \geqslant 2$ have the property that the $\binom{n}{2}$ sums $a_{i}+a_{j}$ are the same as the $\binom{n}{2}$ sums $b_{i}+b_{j}$ in some order. Prove that $n$ is a power of 2 .
Problem 11. Prove the combinatorial identity

$$
\sum_{k=1}^{n} k\binom{n}{k}^{2}=n\binom{2 n-1}{n-1}
$$

Problem 12. Prove the identity

$$
\sum_{k=0}^{m}\binom{m}{k}\binom{n+k}{m}=\sum_{k=0}^{m}\binom{m}{k}\binom{n}{k} 2^{k} .
$$

Problem 13. For $A=\{1,2, \ldots, 100\}$ let $A_{1}, A_{2}, \ldots, A_{m}$ be subsets of $A$ with four elements with the property that any two have at most two elements in common. Prove that if $m \geqslant 40425$ then among these subsets there exist 49 whose union is equal to $A$ but with the union of any 48 of them not equal $A$.
Problem 14. Let $m, n, p, q, r, s$ be positive integers such that $p<r<m$ and $q<s<n$. In how many ways can one travel on a rectangular grid from $(0,0)$ to $(m, n)$ such that at each step one of the coordinates increases by one unit and such that the path avoids the points $(p, q)$ and $(r, s)$ ?
Problem 15. Given a $(2 m+1) \times(2 n+1)$ checkerboard in which the four corners are black squares, show that if one removes any one red square and any two black squares the board can be tiled by $1 \times 2$ dominoes.

Problem 16. Consider a $12 \times 12$ chessboard. If one removes 3 corners, is it possible to tile the board with $1 \times 3$ rectangles?
Problem 17. Partition the non-negative integers into two sets $A$ and $B$ such that every positive integer is expressible by $a+a^{\prime} ; a<a^{\prime} ; a, a^{\prime} \in A$ in the same number of ways as by $b+b^{\prime} ; b<b^{\prime} ; b, b^{\prime} \in B$.
Problem 18. Prove that the number of partitions of an integer into odd positive numbers is equal to the number of partitions into distinct positive integers.
Problem 19. Assume that the points of the plane are each colored red or blue. Prove that one of these colors contains pairs of points at every distance.
Problem 20. The points of the plane are each colored either red, yellow or blue. Prove that there are two points of the same color having mutual distance 1 .
Problem 21. I choose a number between 0 and 15. You ask me 7 yes or no questions. I answer all of them, but am allowed to lie once, although I am not required to. Determine my number.
Problem 22. Prove that at any party, two people have the same number of friends present.

