ALGEBRA

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Problem 1. Let $P(x)$ be a polynomial of odd degree with real coefficients. Show that the equation $P(P(x)) = 0$ has at least as many real roots as the equation $P(x) = 0$, counted without multiplicities.

Problem 2. Let $P(x)$ be a polynomial of degree n. Knowing that

$$P(k) = \frac{k}{k+1}, \qquad k = 0, 1, ..., n,$$

find $P(m)$ for $m > n$.

Problem 3. Find the roots of the polynomial

$$P(x) = x^4 - 6x^3 + 18x^2 - 30x + 25$$

knowing that the sum of two of them is 4.

Problem 4. Let $P(z) = az^4 + bz^3 + cz^2 + dz + e = a(z - r_1)(z - r_2)(z - r_3)(z - r_4)$, where a, b, c, d, e are integers, $a \neq 0$. Show that if $r_1 + r_2$ is a rational number, and if $r_1 + r_2 \neq r_3 + r_4$, then $r_1 r_2$ is rational.

Problem 5. Let $P(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$ be a polynomial of degree $n \geqslant 3$. Knowing that $a_{n-1} = -\binom{n}{1}$, $a_{n-2} = \binom{n}{2}$, and that all roots are real, find the remaining coefficients.

Problem 6. Prove

$$\tan\frac{\pi}{2n+1}\tan\frac{2\pi}{2n+1}\tan\frac{3\pi}{2n+1}\cdots\tan\frac{n\pi}{2n+1} = \sqrt{2n+1}.$$

Problem 7. Prove that the polynomial

$$P(x) = x^{101} + 101x^{100} + 102$$

is irreducible in $\mathbb{Z}[x]$.

Problem 8. Prove for every prime p,

$$P(x) = x^{p-1} + x^{p-2} + ... + x + 1$$

is irreducible in $\mathbb{Z}[x]$.

Problem 9. Prove that for any distinct integers $a_1, ..., a_n$, the polynomial

$$P(x) = (x - a_1)^2(x - a_2)^2...(x - a_n)^2 + 1$$

cannot be written as the product of two non-constant polynomials with integer coefficients.

Problem 10. Prove that the sequence $(\sin n)_n$ is dense in $[-1, 1]$.

Problem 11. Show that infinitely many powers of 2 start with the digit 7.

Problem 12. Given a rectangle, we are allowed to fold it in two or in three, parallel to one side or the other, in order to form a smaller rectangle. Prove that for any $\epsilon > 0$ there are finitely many such operations that produce a rectangle with the ratio of the sides lying in the interval $(1 - \epsilon, 1 + \epsilon)$.

Problem 13. A set of points in the plane is invariant under the reflections across the sides of some given regular pentagon. Prove that the set is dense in the plane.

Problem 14. Let R be a ring with identity with the property that $(xy)^2 = x^2y^2$ for all $x, y \in R$. Show that R is commutative.

Problem 15. Let x and y be elements of a ring with identity and n a positive integer. Prove that if $1 - (xy)^n$ is invertible, then so is $1 - (yx)^n$.

Problem 16. Let

$$f_0(x) = \frac{1}{1-x}, \qquad f_n(x) = f_0(f_{n-1}(x)).$$

Evaluate $f_{1976}(1976)$.

Problem 17. Suppose the polynomial $x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n$ can be factored into

$$(x + r_1)(x + r_2) \cdots (x + r_n),$$

where $r_1, ..., r_n$ are real numbers. Prove $(n-1)a_1^2 \geqslant 2na_2$.

Problem 18. If the roots of the equation

$$a_0x^n - na_1x^{n-1} + \frac{n(n-1)}{1 \cdot 2}a_2x^{n-2} - \cdots + (-1)^n a_n = 0$$

are all positive, show that $a_r a_{n-r} > a_0 a_n$ for all values of r between 1 and $n-1$ inclusive, unless the roots are all equal.

Problem 19. Given any n real pairs (x_i, y_i) with the x_i all distinct, prove that the interpolation problem $P(x_i) = y_i$ can be solved by a polynomial P, all of whose zeros are real.

Problem 20. $x_{n+1} = \frac{x_n + x_{n-1}}{2}$. Find $\lim x_n$ as a function of x_0 and x_1.

Problem 21. A fair way of splitting a pie between two people allows the first to cut the pie, and the second to choose which piece he prefers. How would you fairly split a pie among three people?