SEQUENCES AND SERIES

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Problem 1. Find the formula for the general term of the sequence

$$1, 2, 2, 3, 3, 3, 4, 4, 4, 4, 5, 5, 5, 5, 5, ...$$

Problem 2. Let $a = 4k - 1$, where k is an integer. Prove that for any positive integer n the number

$$1 - \binom{n}{2} a + \binom{n}{4} a^2 - \binom{n}{6} a^3 + \cdots$$

is divisible by 2^{n-1}.

Problem 3. Let A and E be opposing vertices of a regular octagon. A frog starts jumping at vertex A. From any vertex of the octagon except E, it may jump to either of the two adjacent vertices. When it reaches vertex E, the frog stops and stays there. Let a_n be the number of distinct paths of exactly n jumps ending at E. Prove that $a_{2n-1} = 0$ and

$$a_{2n} = \frac{1}{\sqrt{2}}(x^{n-1} + y^{n-1}), \quad n = 1, 2, 3, ...,$$

where $x = 2 + \sqrt{2}$ and $y = 2 - \sqrt{2}$.

Problem 4. Compute $\lim_{n\to\infty} |\sin(\pi\sqrt{n^2 + n + 1})|$.

Problem 5. Let k be a positive integer and μ a positive real number. Prove that

$$\lim_{n\to\infty} \binom{n}{k} \left(\frac{\mu}{n}\right)^k \left(1 - \frac{\mu}{n}\right)^{n-k} = \frac{\mu^k}{e^{\mu} k!}.$$

Problem 6. Let $(x_n)_n$ be a sequence of positive integers such that $x_{x_n} = n^4$ for all $n \geqslant 1$. Is it true that $\lim_{n\to\infty} x_n = \infty$?

Problem 7. Compute

$$\lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{k}{n^2}\right)^{\frac{k}{n^2}+1}.$$

Problem 8. Prove that the sequence $(a_n)_{n\geqslant 1}$ defined by

$$a_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln(n+1), \quad n \geqslant 1,$$

is convergent.

Problem 9. Compute

$$\sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \cdots}}}}.$$

Problem 10. Let $P(x) = a_m x^m + a_{m-1}x^{m-1} + \cdots + a_0$, $a_i > 0$, $i = 0, 1, 2, ..., m$. Denote by A_n and G_n the arithmetic and geometric means of $P(1), P(2), ..., P(n)$. Prove that

$$\lim_{n\to\infty} \frac{A_n}{G_n} = \frac{e^m}{m+1}.$$

Problem 11. Let q range over numbers n^m with $n, m \geqslant 2$. Prove

$$\sum_q \frac{1}{q-1} = 1.$$

Problem 12. Evaluate in closed form

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{m!n!}{(m+n+2)!}.$$

Problem 13. Let x be a real number. Define the sequence $(x_n)_{n \geqslant 1}$ recursively by $x_1 = 1$ and $x_{n+1} = x^n + nx_n$ for $n \geqslant 1$. Prove that

$$\prod_{n=1}^{\infty} \left(1 - \frac{x^n}{x_{n+1}}\right) = e^{-x}.$$

Problem 14. Let ζ be a root of unity. Prove that

$$\zeta^{-1} = \sum_{n=0}^{\infty} \zeta^n (1-\zeta)(1-\zeta^2) \cdots (1-\zeta^n)$$

with the convention that the zeroth term is 1.