REAL ANALYSIS

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Problem 1. Let $f : (0,\infty) \to (0,\infty)$ be an increasing function with $\lim_{t\to\infty} \frac{f(2t)}{f(t)} = 1$. Prove that $\lim_{t\to\infty} \frac{f(mt)}{f(t)} = 1$ for any $m > 0$.

Problem 2. Let $f(x) = \sum_{k=1}^n a_k \sin kx$, with $a_1, a_2, ..., a_n \in \mathbb{R}$, $n \geqslant 1$. Prove that if $f(x) \leqslant |\sin x|$ for all $x \in \mathbb{R}$, then
$$\left| \sum_{k=1}^n k a_k \right| \leqslant 1.$$

Problem 3. Let $f : [a,b] \to [a,b]$ be continuous. Prove that f has a fixed point.

Problem 4. Prove that any convex polygonal surface may be divided by two perpendicular lines into four regions of equal area.

Problem 5. Let $f : I \to \mathbb{R}$ be a function defined on an interval. Show that if f has the intermediate value property and for any $y \in \mathbb{R}$ the set $f^{-1}(y)$ is closed, then f is continuous.

Problem 6. Determine
$$\max_{z\in\mathbb{C}:|z|=1} |z^3 - z + 2|.$$

Problem 7. Show that if a function $f : [a,b] \to \mathbb{R}$ is convex, then it is continuous on (a,b).

Problem 8. Let $0 < a < b$ and $t_i \geqslant 0$, $i = 1, 2, ..., n$. Prove that for any $x_1, x_2, ..., x_n \in [a,b]$,
$$\left(\sum_{i=1}^n t_i x_i\right)\left(\sum_{i=1}^n \frac{t_i}{x_i}\right) \leqslant \frac{(a+b)^2}{4ab}\left(\sum_{i=1}^n t_i\right)^2.$$

Problem 9. Prove that for any $n \geqslant 2$ and any $|x| \leqslant 1$,
$$(1+x)^n + (1-x)^n \leqslant 2^n.$$

Problem 10. Let $0 < x_i < \pi$, $i = 1, 2, ..., n$, and set $x = \frac{x_1 + \cdots + x_n}{n}$. Prove that
$$\prod_{i=1}^n \left(\frac{\sin x_i}{x_i}\right) \leqslant \left(\frac{\sin x}{x}\right)^n.$$

Problem 11. Compute
$$\int_0^{\frac{\pi}{4}} \ln(1 + \tan x) dx.$$

Problem 12. Find
$$\int_0^1 \frac{\ln(1+x)}{1+x^2} dx.$$

Problem 13. Let $n \geqslant 0$ be an integer. Compute the integral
$$\int_0^\pi \frac{1 - \cos nx}{1 - \cos x} dx.$$

Problem 14. Compute

$$\lim_{n\to\infty}\left(\frac{2^{\frac{1}{n}}}{n+1}+\frac{2^{\frac{2}{n}}}{n+\frac{1}{2}}+\cdots+\frac{2^{\frac{n}{n}}}{n+\frac{1}{n}}\right).$$

Problem 15. Let $f:[0,1]\to\mathbb{R}$ be a continuous function such that

$$\int_0^1 f(x)dx=\int_0^1 xf(x)dx=1.$$

Prove that

$$\int_0^1 f^2(x)dx\geqslant 4.$$

Problem 16. Let $f:[0,\infty)\to[0,\infty)$ be a continuous, strictly increasing function with $f(0)=0$. Prove that

$$\int_0^a f(x)dx+\int_0^b f^{-1}(x)dx\geqslant ab$$

for all positive numbers a,b with equality if and only if $b=f(a)$. Here f^{-1} denotes the inverse function of f.

Problem 17. Let $f:[0,1]\to[0,\infty)$ be a differentiable function with decreasing first derivative, and such that $f(0)=0$ and $f'(0)>0$. Prove that

$$\int_0^1\frac{dx}{f^2(x)+1}\leqslant\frac{f(1)}{f'(1)}.$$

Can equality hold?

Problem 18. Compute the ratio

$$\frac{1+\frac{\pi^4}{5!}+\frac{\pi^8}{9!}+\frac{\pi^{12}}{13!}+\cdots}{\frac{1}{3!}+\frac{\pi^4}{7!}+\frac{\pi^8}{11!}+\frac{\pi^{12}}{15!}+\cdots}.$$

Problem 19. Prove that for $|x|<1$,

$$\arcsin x=\sum_{k=0}^{\infty}\frac{1}{2^{2k}(2k+1)}\binom{2k}{k}x^{2k+1}.$$

Problem 20.

a. Prove that for $|x|<2$,

$$\sum_{k=1}^{\infty}\frac{x^{2k}}{\binom{2k}{k}}=\frac{x\left(4\arcsin\left(\frac{x}{2}\right)+x\sqrt{4-x^2}\right)}{(4-x^2)\sqrt{4-x^2}}.$$

b. Prove the identity

$$\sum_{k=1}^{\infty}\frac{1}{\binom{2k}{k}}=\frac{2\pi\sqrt{3}+36}{27}.$$

Problem 21. Prove that for every $0<x<2\pi$,

$$\frac{\pi-x}{2}=\frac{\sin x}{1}+\frac{\sin 2x}{2}+\frac{\sin 3x}{3}+\cdots.$$

Thus

$$\frac{\pi}{4}=\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{2k-1}.$$