POLYNOMIALS

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Over the complex numbers the fundamental theorem of algebra states that a polynomial factors into linear factors $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0 = a_n(x - x_1)\cdots(x - x_n)$. The Viète relations state that the elementary symmetric polynomials in the roots can be obtained from the coefficients

$$\begin{aligned} x_1 + x_2 + \cdots + x_n &= -\frac{a_{n-1}}{a_n} \\ x_1x_2 + x_1x_3 + \cdots + x_{n-1}x_n &= \frac{a_{n-2}}{a_n} \\ &\vdots \\ x_1x_2\cdots x_n &= (-1)^n\frac{a_0}{a_n}. \end{aligned}$$

Theorem 1 (Gauss-Lucas Theorem)**.** *The zeros of the derivative $P'(z)$ of a complex polynomial $P(z)$ lie within the convex hull of the roots.*

Proof. After rotation and translation, assume all of the roots of $P(z)$ lie above the real axis. If $\mathrm{Re}(w) < 0$ then

$$\mathrm{Im}\,\frac{P'(w)}{P(w)} = \sum_{k=1}^{n} \mathrm{Im}\,\frac{1}{w - z_k} > 0.$$

Thus the roots of $P'(z)$ are all above the real axis. □

Theorem 2 (Eisenstein's criterion)**.** *Given a polynomial $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$ with integer coefficients, suppose that there exists a prime number p such that a_n is not divisible by p, a_k is divisible by p for $k = 0, 1, ..., n-1$, and a_0 is not divisible by p^2. Then $P(x)$ is irreducible over $\mathbb{Z}[x]$.*

Proof. Suppose $P(x) = Q(x)R(x)$, with $Q(x)$, $R(x)$ not identically ± 1. Let

$$\begin{aligned} Q(x) &= b_k x^k + b_{k-1}x^{k-1} + \cdots + b_0, \\ R(x) &= c_{n-k}x^{n-k} + c_{n-k-1}x^{n-k-1} + \cdots + c_0. \end{aligned}$$

Then one of b_0, c_0 is divisible by p, say b_0 is divisible by p but c_0 is not. It now follows that all of the other b_k are divisible by p from induction, by reducing the equation

$$a_k = b_k c_0 + b_{k-1}c_1 + ... + b_0 c_k.$$

This is a contradiction. □

Problem 1. Find a polynomial with integer coefficients and root $\sqrt{2} + \sqrt{3}$.

Problem 2. Let $P(x) = x^4 + ax^3 + bx^2 + cx + d$ and $Q(x) = x^2 + px + q$ be two polynomials with real coefficients. Suppose that there exists an interval (r, s) of length greater than 2 such that both $P(x)$ and $Q(x)$ are negative for $x \in (r, s)$ and both are positive for $x < r$ or $x > s$. Show that there is a real number x_0 such that $P(x_0) < Q(x_0)$.

Problem 3. Let $P(x)$ be a polynomial of degree n. Given $P(k) = \frac{k}{k+1}$, $k = 0, 1, ..., n$, find $P(m)$ for $m > n$.

Problem 4. Find all polynomials whose coefficients are equal either to 1 or -1 and whose zeros are all real.

Problem 5. Let $P(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$ be a polynomial of degree $n \geqslant 3$. Knowing that $a_{n-1} = -\binom{n}{1}$, $a_{n-2} = \binom{n}{2}$, and that all the roots are real, find the remaining coefficients.

Problem 6. Let $P(z)$ and $Q(z)$ be polynomials with complex coefficients of degree greater than or equal to 1 with the property that $P(z) = 0$ if and only if $Q(z) = 0$ and $P(z) = 1$ if and only if $Q(z) = 1$. Prove that the polynomials are equal.

Problem 7. Let $P_n(x) = (x^n - 1)(x^{n-1} - 1) \cdots (x - 1)$, $n \geqslant 1$. Prove that for $n \geqslant 2$, $P_n'(x)$ is divisible by $P_{\lfloor \frac{n}{2} \rfloor}(x)$ in the ring of polynomials with integer coefficients.

Problem 8. Let $P(x)$ be a polynomial of degree $n > 3$ whose zeros $x_1 < x_2 < \cdots < x_{n-1} < x_n$ are real. Prove that

$$P'\left(\frac{x_1 + x_2}{2}\right) P'\left(\frac{x_{n-1} + x_n}{2}\right) \neq 0.$$

Problem 9. For $a \neq 0$ a real number and $n > 2$ an integer, prove that every nonreal root z of the polynomial equation $x^n + ax + 1 = 0$ satisfies the inequality $|z| \geqslant \left(\frac{1}{n-1}\right)^{\frac{1}{n}}$.

Problem 10. Prove that for any distinct integers $a_1, a_2, ..., a_n$ the polynomial

$$P(x) = (x - a_1)^2(x - a_2)^2 \cdots (x - a_n)^2 + 1$$

cannot be written as the product of two non-constant polynomials with integer coefficients.

Problem 11. Associate to a prime the polynomial whose coefficients are the decimal digits of the prime (for example, 7043 is associated to $P(z) = 7z^3 + 4z + 3$). Prove that this polynomial is always irreducible over $\mathbb{Z}[z]$.

Problem 12. Let p be a prime number. Prove that the polynomial

$$P(x) = x^{p-1} + 2x^{p-2} + 3x^{p-3} + \cdots + (p-1)x + p$$

is irreducible in $\mathbb{Z}[x]$.

Problem 13. Let $P(x)$ be a monic polynomial in $\mathbb{Z}[x]$, irreducible over this ring, and such that $|P(0)|$ is not the square of an integer. Prove that the polynomial $Q(x)$ defined by $Q(x) = P(x^2)$ is also irreducible over $\mathbb{Z}[x]$.

Problem 14. Find polynomials $F(x)$ and $G(x)$ such that

$$(x^8 - 1)F(x) + (x^5 - 1)G(x) = x - 1.$$

Problem 15. Show that each polynomial $(\cos\theta + x\sin\theta)^n - \cos n\theta - x\sin n\theta$ is divisible by $x^2 + 1$.

Problem 16.

a. Suppose $f(x)$ is a polynomial over the real numbers and $g(x)$ is a divisor of $f(x)$ and $f'(x)$ which is irreducible or square free. Show that $(g(x))^2$ divides $f(x)$.
b. Factor $x^6 + x^4 + 3x^2 + 2x + 2$ into a product of irreducibles over the complex numbers.

Problem 17. Let $x^{(n)} = x(x-1)\cdots(x-n+1)$ for n a positive integer, and let $x^{(0)} = 1$. Prove that for all real numbers x and y

$$(x+y)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} x^{(k)} y^{(n-k)}.$$