NUMBER THEORY

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Problem 1. Let k be a positive integer. The sequence $(a_n)_n$ is defined by $a_1 = 1$, and for $n \geqslant 2$, a_n is the nth positive integer greater than a_{n-1} that is congruent to n modulo k. Find a_n in closed form.

Problem 2. Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ with the property that

$$2f(f(x)) - 3f(x) + x = 0, \qquad x \in \mathbb{Z}.$$

Problem 3. Prove that there exists no bijection $f : \mathbb{N} \to \mathbb{N}$ such that

$$f(mn) = f(m) + f(n) + 3f(m)f(n),$$

for all $m, n \geqslant 1$.

Problem 4. Prove that the system of equations

$$x^2 + 5y^2 = z^2,$$
$$5x^2 + y^2 = t^2$$

does not admit nontrivial integer solutions.

Problem 5. Show that the equation

$$x^2 - y^2 = 2xyz$$

has no solutions in the set of positive integers.

Problem 6. Prove the identity

$$\sum_{k=1}^{\frac{n(n+1)}{2}} \left\lfloor \frac{-1 + \sqrt{1 + 8k}}{2} \right\rfloor = \frac{n(n^2+2)}{3}, \quad n \geqslant 1.$$

Problem 7. Prove that for any real number x and for any positive integer n,

$$\lfloor nx \rfloor \geqslant \frac{\lfloor x \rfloor}{1} + \frac{\lfloor 2x \rfloor}{2} + \cdots + \frac{\lfloor nx \rfloor}{n}.$$

Problem 8. Show that for any positive integer n,

$$\lfloor \sqrt{n} \rfloor = \left\lfloor \sqrt{n} + \frac{1}{\sqrt{n} + \sqrt{n+2}} \right\rfloor.$$

Problem 9. Prove that for $a, b \geqslant 1$,

$$\mathrm{GCD}(n^a - 1, n^b - 1) = n^{\mathrm{GCD}(a,b)} - 1.$$

Problem 10. Determine the functions $f : \{0, 1, 2, ...\} \to \{0, 1, 2, ...\}$ satisfying

a. $f(2n+1)^2 - f(2n)^2 = 6f(n) + 1$
b. $f(2n) \geqslant f(n)$ for all $n \geqslant 0$.

Problem 11. Let p be a prime number. Prove that there are infinitely many multiples of p whose last ten digits are all distinct.

Problem 12. Find all n such that $n!$ ends in exactly 1000 zeros.

Problem 13. Prove that $\frac{\mathrm{GCD}(m,n)}{n}\binom{n}{m}$ is an integer for all pairs of integers $n \geqslant m \geqslant 1$.

Problem 14. Prove that if n is a positive integer that is divisible by at least two primes, then there is an n-gon with all angles equal and with side lengths the numbers $1, 2, ..., n$ in some order.

Problem 15. Let $f(x_1, x_2, ..., x_n)$ be a polynomial with integer coefficients of total degree less than n. Show that the number of ordered n-tuples $(x_1, x_2, ..., x_n)$ with $0 \leqslant x_i \leqslant 12$ such that $f(x_1, ..., x_n) \equiv 0 \bmod 13$ is divisible by 13.

Problem 16. Let p be an odd prime and $a_1, a_2, ..., a_p$ an arithmetic progression whose common difference is not divisible by p. Prove that there exists an index i such that the number $a_1 a_2 \cdots a_p + a_i$ is divisible by p^2.

Problem 17. Let a, b, c, d be integers with the property that for any two integers m and n there exist integers x and y satisfying the system

$$\begin{aligned} ax + by &= m, \\ cx + dy &= n. \end{aligned}$$

Prove that $ad - bc = \pm 1$.

Problem 18. Given a piece of paper, we can cut it into 8 or 12 pieces. Any of these pieces can be cut into 8 or 12 pieces, and so on. Show that we can obtain any number of pieces greater than 60. Can 60 pieces be obtained?

Problem 19. Find a solution to the Diophantine equation

$$x^2 - (m^2 + 1)y^2 = 1.$$

Problem 20. Prove that the equation

$$x^3 + y^3 + z^3 + t^3 = 1999$$

has infinitely many integers solutions.

Problem 21. Prove that every pair of positive integers m and n, there exists a positive integer p satisfying

$$(\sqrt{m} + \sqrt{m-1})^n = \sqrt{p} + \sqrt{p-1}.$$