MULTIVARIABLE ANALYSIS

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The main theorems of multivariable analysis are as follows.

Theorem 1 (Lagrange multipliers)**.** *If a smooth function $f(x)$ subject to the smooth constraint $g(x) = c$ has a maximum or minimum, it occurs where the gradients of f and g are parallel.*

Theorem 2 (Green's theorem)**.** *Let D be a domain in the plane with boundary C oriented such that D is to the left. If the vector field $F(x, y) = \begin{pmatrix} P(x,y) \\ Q(x,y) \end{pmatrix}$ is continuously differentiable on D, then*

$$\oint\limits_C Pdx + Qdy = \iint\limits_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy.$$

Theorem 3 (Stokes' theorem)**.** *Let M be a smooth, oriented n-manifold with boundary ∂M and let ω be a differential $n-1$-form. Then*

$$\int_{\partial M} \omega = \int_M d\omega$$

where d is the exterior derivative.

In three dimensions this takes several forms.

Theorem 4 (Classical Stokes' theorem)**.** *Let S be an oriented surface with normal vector N, bounded by a closed, piecewise smooth curve C that is oriented such that if one travels on C with the upward direction N, the surface is on the left. If F is a vector field that is continuously differentiable on S, then*

$$\oint\limits_C F \cdot dR = \iint\limits_S \mathrm{curl} F \cdot N dS.$$

Gauss's theorem is also implied by Stokes' theorem.

Theorem 5 (Gauss's divergence theorem)**.** *Let S be a smooth, orientable surface that encloses a solid region V in space. If F is a continuously differentiable vector field on V, then*

$$\iint\limits_S F \cdot N dS = \iiint\limits_V \div F dV.$$

Theorem 6 (Cauchy's theorem)**.** *Let Γ be an oriented curve that bounds a region Δ on its left and let $a \in \Delta$. If $f(z)$ is analytic in Δ and continuous on the boundary, then*

$$\frac{f^{(n)}(a)}{n!} = \frac{1}{2\pi i} \int_\Gamma \frac{f(z)}{(z-a)^{n+1}} dz.$$

Problem 1. Prove that the function $u(x, t)$ satisfies the equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad (x, t) \in \mathbb{R}^2$$

then so does the function

$$v(x,t) = \frac{1}{\sqrt{t}} e^{-\frac{x^2}{4t}} u(xt^{-1}, -t^{-1}), \quad x \in \mathbb{R}, \quad t > 0.$$

Problem 2. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a differentiable function with continuous partial derivatives and with $f(0,0) = 0$. Prove that there exist continuous functions $g_1, g_2 : \mathbb{R}^2 \to \mathbb{R}$ such that

$$f(x,y) = xg_1(x,y) + yg_2(x,y).$$

Problem 3. Given n points in the plane, suppose there is a unique line that minimizes the sum of the distances from the points to the line. Prove that the line passes through two of the points.

Problem 4. Compute the integral $\iint_D x dx dy$, where

$$D = \left\{ (x,y) \in \mathbb{R}^2 : x \geqslant 0, 1 \leqslant xy \leqslant 2, 1 \leqslant \frac{y}{x} \leqslant 2 \right\}.$$

Problem 5. Compute the integral

$$\iint\limits_D \frac{dxdy}{(x^2+y^2)^2}$$

where D is the domain bounded by the circles

$$x^2 + y^2 - 2x = 0, \quad x^2 + y^2 - 4x = 0,$$
$$x^2 + y^2 - 2y = 0, \quad x^2 + y^2 - 6y = 0.$$

Problem 6. Evaluate

$$\int_0^1 \int_0^1 \int_0^1 (1 + u^2 + v^2 + w^2)^{-2} du dv dw.$$

Problem 7. Let $|x| < 1$. Prove that

$$\sum_{n=1}^{\infty} \frac{x^n}{n^2} = -\int_0^x \frac{1}{t} \ln(1-t) dt.$$

Problem 8. Let $\phi(x,y,z)$ and $\psi(x,y,z)$ be twice continuously differentiable functions in the region $\{(x,y,z) : \frac{1}{2} < \sqrt{x^2+y^2+z^2} < 2\}$. Prove that

$$\iint\limits_S (\nabla\phi \times \nabla\psi) \cdot \vec{n} dS = 0,$$

where S is the unit sphere centered at the origin, $\vec{n}$ is the normal unit vector to this sphere, and $\nabla\phi$ is the gradient.

Problem 9. Let $f, g : \mathbb{R}^3 \to \mathbb{R}$ be twice continuously differentiable functions that are constant along the lines that pass through the origin. Prove that on the unit ball B,

$$\iiint\limits_B f \Delta g dV = \iiint\limits_B g \Delta f dV$$

where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$.

Problem 10. Determine all continuous functions $\phi, f, g, h : \mathbb{R} \to \mathbb{R}$ satisfying the condition

$$\phi(x+y+z) = f(x) + g(y) + h(z),$$

for all real $x, y, z \in \mathbb{R}$.

Problem 11. For $n \geqslant 2$, determine those continuous functions $f : \mathbb{R} \to \mathbb{R}$ with the property that for all real $x_1, x_2, ..., x_n$,

$$\sum_i f(x_i) - \sum_{i<j} f(x_i + x_j) + \sum_{i<j<k} f(x_i + x_j + x_k) + \cdots + (-1)^{n-1} f(x_1 + \cdots + x_n) = 0.$$

Problem 12. Find those continuously differentiable functions $f, g : \mathbb{R} \to \mathbb{R}$ satisfying

$$f^2 + g^2 = f'^2 + g'^2, \quad f + g = g' - f',$$

and such that $f = g$ has two real solutions, the smaller being 0.

Problem 13. Find all continuously differentiable functions $y : (0, \infty) \to (0, \infty)$ that are solutions to the initial value problem

$$y^{y'} = x, \quad y(1) = 1.$$

Problem 14. Prove that if the function $f(x, y)$ is continuously differentiable on the whole xy-plane and satisfies the equation

$$\frac{\partial f}{\partial x} + f \frac{\partial f}{\partial y} = 0,$$

then $f(x, y)$ is constant.

Problem 15. Solve the differential equation

$$xy'' + 2y' + xy = 0.$$

Problem 16. Let n be a positive integer. Show that the equation

$$(1 - x^2)y'' - xy' + n^2 y = 0$$

admits as a particular solution an nth degree polynomial.

Problem 17. Show that both functions

$$y_1(x) = \int_0^\infty \frac{e^{-tx}}{1 + t^2} dt \quad \text{and} \quad y_2(x) = \int_0^\infty \frac{\sin t}{t + x} dt$$

satisfy the differential equation $y'' + y = \frac{1}{x}$. Prove that these two functions are equal.

Problem 18. Find the one-to-one, twice-differentiable solutions y to the equation

$$\frac{d^2 y}{dx^2} + \frac{d^2 x}{dy^2} = 0.$$

Problem 19. Let f be a real-valued continuous nonnegative function on $[0, 1]$ such that

$$f(t)^2 \leqslant 1 + 2 \int_0^t f(s) ds, \qquad t \in [0, 1].$$

Show that $f(t) \leqslant 1 + t$ for every $t \in [0, 1]$.

Problem 20. Determine all nth degree polynomials $P(x)$ with real zeros, for which the equality

$$\sum_{i=1}^{n} \frac{1}{P(x) - x_i} = \frac{n^2}{xP'(x)}$$

holds for all nonzero real numbers x for which $P'(x) \neq 0$, where x_i, $i = 1, 2, ..., n$ are the roots of $P(x)$.