LINEAR ALGEBRA AND ABSTRACT ALGEBRA

ROBERT HOUGH

Problem 1. Let A and B be 2×2 matrices with real entries satisfying $(AB - BA)^n = I$ for some positive integer n. Prove that n is even, and $(AB - BA)^4 = I$.

Problem 2. Let a, b, c, d be real numbers, $c \neq 0$ with $ad - bc = 1$. Prove that there exist u and v such that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & -u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} \begin{pmatrix} 1 & -v \\ 0 & 1 \end{pmatrix}.$$

Problem 3. Let A and B be $n \times n$ matrices with real entries such that

$$\mathrm{tr}(AA^t + BB^t) = \mathrm{tr}(AB + A^t B^t).$$

Prove that $A = B^t$.

Problem 4. Prove

$$\det \begin{pmatrix} (x^2+1)^2 & (xy+1)^2 & (xz+1)^2 \\ (xy+1)^2 & (y^2+1)^2 & (yz+1)^2 \\ (xz+1)^2 & (yz+1)^2 & (z^2+1)^2 \end{pmatrix} = 2(y-z)^2(z-x)^2(x-y)^2.$$

Problem 5. Given distinct integers $x_1, x_2, ..., x_n$, prove that $\prod_{i<j}(x_i - x_j)$ is divisible by $1!2! \cdots (n-1)!$.

Problem 6. Let A and B be 3×3 matrices with real elements such that $\det A = \det B = \det(A+B) = \det(A-B) = 0$. Prove that $\det(xA + yB) = 0$ for any real numbers x and y.

Problem 7. Let A, B, C, D be $n \times n$ matrices such that $AC = CA$. Prove that

$$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(AD - CB).$$

Problem 8. Let $P(t)$ be a polynomial of even degree with real coefficients. Prove that the function $f(X) = P(X)$ defined on the set of $n \times n$ matrices is not onto.

Problem 9. Let n be an odd positive integer and A and $n \times n$ matrix with the property that $A^2 = 0$ or $A^2 = I$. Prove that $\det(A + I) \geqslant \det(A - I)$.

Problem 10. Let A be an $n \times n$ matrix such that, for each i, $\sum_{j=1}^n |A_{ij}| < 1$ for each i. Prove that $I - A$ is invertible.

Problem 11. Solve the system of linear equations

$$\begin{aligned} x_1 + x_2 + x_3 &= 0 \\ x_2 + x_3 + x_4 &= 0 \\ &\vdots \\ x_{100} + x_1 + x_2 &= 0. \end{aligned}$$

Problem 12. Let $P(x) = x^n + x^{n-1} + \cdots + x + 1$. Find the remainder obtained when $P(x^{n+1})$ is divided by $P(x)$.

Problem 13. Let n be a positive integer and $P(x)$ an nth degree polynomial with complex coefficients such that $P(0), P(1), ..., P(n)$ are all integers. Prove that the polynomial $n!P(x)$ has integer coefficients.

Problem 14. For integers $n \geqslant 2$ and $0 \leqslant k \leqslant n-2$, compute the determinant

$$\det \begin{pmatrix} 1^k & 2^k & 3^k & \cdots & n^k \\ 2^k & 3^k & 4^k & \cdots & (n+1)^k \\ 3^k & 4^k & 5^k & \cdots & (n+2)^k \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ n^k & (n+1)^k & (n+2)^k & \cdots & (2n-1)^k \end{pmatrix}.$$

Problem 15. Let V be a vector space and let $f, f_1, f_2, ..., f_n$ be linear maps from V to $\mathbb{R}$. Suppose that $f(x) = 0$ whenever $f_1(x) = f_2(x) = \cdots = f_n(x) = 0$. Prove that f is a linear combination of $f_1, f_2, ..., f_n$.

Problem 16. Let U and V be isometric linear transformations of $\mathbb{R}^n$, $n \geqslant 1$, with the property that $\|Ux - x\| \leqslant \frac{1}{2}$ and $\|Vx - x\| \leqslant \frac{1}{2}$ for all $x \in \mathbb{R}^n$ with $\|x\| = 1$. Prove that

$$\|UVU^{-1}V^{-1}x - x\| \leqslant \frac{1}{2},$$

for all $x \in \mathbb{R}^n$ with $\|x\| = 1$.

Problem 17. Let A be a square matrix whose off-diagonal entries are positive. Prove that the right-most eigenvalue of A in the complex plane is real and all other eigenvalues are strictly to the left in the complex plane.

Problem 18. Prove that $(\sin n)_n$ is dense in the interval $[-1, 1]$.

Problem 19. Show that infinitely many powers of 2 start with the digit 7.

Problem 20. Let R be a nontrivial ring with identity, and $M = \{x \in R : x = x^2\}$ the set of its idempotents. Prove that if M is finite, then it has an even number of elements.

Problem 21. Let R be a ring with identity such that $x^6 = x$ for all $x \in R$. Prove that $x^2 = x$ for all $x \in R$. Prove that any such ring is commutative.