INDUCTION, PIGEONHOLE AND EXTREMAL CONFIGURATIONS

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To prove a statement $P(n)$ for all positive integers n by induction, prove that $P(1)$ is true, and prove that $P(n)$ implies $P(n+1)$. Consider the following example:

Theorem 1. *Every integer $n \geqslant 1$ can be written $F_{k_1} + F_{k_2} + \cdots + F_{k_r}$ where the F_i are Fibonacci numbers and $k_i \geqslant k_{i+1} + 2$.*

Proof. Base case: 1 is a Fibonacci number.

Inductive step: Assume by induction that $n \geqslant 1$ has a representation $F_{k_1}+F_{k_2}+\cdots+F_{k_r}$ as above. Then $n+1 = F_{k_1} + F_{k_2} + \cdots + F_{k_r} + 1$. If $F_{k_r} \geqslant 3$ we are done, since 1 is a Fibonacci number. If F_{k_r} is 1 or 2, replace F_{k_r} with $F_{k_r} + 1 = F_{k_r+1}$, which is again a Fibonacci number. It may now be the case that $k_r + 1 = k_{r-1} - 1$. If this is the case, replace $F_{k_r+1} + F_{k_{r-1}}$ with $F_{k_{r-1}+1}$. Continue doing so until it is no longer possible to replace the last two terms with their sum. □

The pigeonhole principle states that a function $f : S \to T$ from a finite set S to a finite set T with $|S| > |T|$ has some $t \in T$ with $|f^{-1}(t)| \geqslant 2$.

Theorem 2. *There are integers a, b, c, each of absolute value less than 1 million, such that such that $|a + b\sqrt{2} + c\sqrt{3}| < 10^{-11}$.*

Proof. Let S be the set of 10^{18} real numbers $a + b\sqrt{2} + c\sqrt{3}$ with $0 \leqslant a, b, c, < 10^6$. These numbers are all in the interval $[0, (1 + \sqrt{2} + \sqrt{3})10^6]$. Partition this interval into $10^{18} - 1$ equal size subintervals, each of length $\frac{(1+\sqrt{2}+\sqrt{3})10^6}{10^{18}-1} < 10^{-11}$. By the pigeonhole principle, two members of S lie in the same interval, and their difference gives the claimed sum. □

When trying to prove a theorem, it can help to consider the extreme case. Consider the following example.

Theorem 3. *It is not possible to partition a cube into finitely many smaller cubes of distinct sizes.*

Proof. Suppose for contradiction that such a partition exists. Consider the bottom face of the dissected cube, which is a square dissected into smaller squares of differing edge length. Pick the smallest of these. It is easy to see that this square may not touch an edge, by considering the squares adjacent to it. Thus the square is surrounded on each side by squares of larger side length. Consider the cube which has this square as a bottom square. The top face of this cube is again partitioned into smaller squares. The argument can then be repeated ad absurdum, since the smallest cube above a square never reaches the top. □

Problem 1. Prove that for each $n \geqslant 2$, there exists $k \in \mathbb{N}$ such that k can be written as a sum of i non-zero squares for each $2 \leqslant i \leqslant n$.

Problem 2. Prove for all positive integers the identity

$$\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} = 1 - \frac{1}{2} + \frac{1}{3} - \cdots + \frac{1}{2n-1} - \frac{1}{2n}.$$

Problem 3. Prove that $|\sin nx| \leqslant n|\sin x|$ for any real number x and positive integer n.

Problem 4. Prove that for any positive integer n there exists an n-digit number

(1) divisible by 2^n and containing only the digits 2 and 3,
(2) divisible by 5^n and containing only the digits $5, 6, 7, 8, 9$.

Problem 5. Given a sequence of integers $x_1, x_2, ..., x_n$ whose sum is 1, prove that exactly one of the cyclic shifts

$$x_1, x_2, ..., x_n; \quad x_2, x_3, ..., x_n, x_1; \quad ...; \qquad x_n, x_1, ..., x_{n-1}$$

has all of its partial sums positive.

Problem 6. The vertices of a convex polygon are colored by at least three colors such that no two consecutive vertices have the same color. Prove that one can dissect the polygon into triangles by diagonals that do not cross and whose endpoints have different colors.

Problem 7. Show that if $a_1, ..., a_n$ are non-negative numbers, then

$$(1 + a_1)(1 + a_2) \cdots (1 + a_n) \geqslant (1 + (a_1...a_n)^{\frac{1}{n}})^n.$$

Problem 8. A sequence of m positive integers contains exactly n distinct terms. Prove that if $2^n \leqslant m$ then there exists a block of consecutive terms whose product is a perfect square.

Problem 9. Let p be prime and let a, b, c be integers such that a and b are not divisible by p. Prove that $ax^2 + by^2 \equiv c \bmod p$ has integer solutions.

Problem 10. A chess player trains by playing at least one game per day, but, to avoid exhaustion, no more than 12 games a week. Prove that there is a group of consecutive days in which he plays exactly 20 games.

Problem 11. Let $x_1, x_2, ..., x_k$ be real numbers such that the set $A = \{\cos(n\pi x_1) + \cos(n\pi x_2) + \cdots + \cos(n\pi x_k) | n \geqslant 1\}$ is finite. Prove that the x_i are all rational.

Problem 12. Given 21 points in the plane, no three on a line, draw the lines connecting two points. Prove that two intersect making an angle of less than 1 degree.

Problem 13. The points of the plane are colored by finitely many colors. Prove that one can find a rectangle with vertices of the same color.

Problem 14. Given $n \geqslant 3$ points in the plane, prove that some three of them form an angle less than or equal to $\frac{\pi}{n}$.

Problem 15. Consider a planar region of area 1, obtained as the union of finitely many disks. Prove that from these disks we can select some that are mutually disjoint and have total area at least $\frac{1}{9}$.

Problem 16. Prove that among any eight distinct positive integers less than 2004 there are four, say a, b, c and d, such that

$$4 + d \leqslant a + b + c \leqslant 4d.$$

Problem 17. Let $a_1, a_2, ..., a_n, ...$ be a sequence of distinct positive integers. Prove that for any positive integer n,

$$a_1^2 + ... + a_n^2 \geqslant \frac{2n + 1}{3}(a_1 + ... + a_n).$$

Problem 18. The positive integers are colored by two colors. Prove that there exists an infinite sequence of positive integers $k_1 < k_2 < k_3 < ...$ with the property that $2k_1 < k_1 + k_2 < 2k_2 < k_2 + k_3 < ...$ all have the same color.