GEOMETRY AND TRIGONOMETRY

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Problem 1. Consider three linearly independent vectors $\vec{a}, \vec{b}, \vec{c}$ in space, having the same origin. Prove that the plane determined by the endpoints of the vectors is perpendicular to the vector $\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a}$.

Problem 2. Suppose $\vec{a}, \vec{b}, \vec{c}$ satisfy

$$\vec{a} \times \vec{b} = \vec{b} \times \vec{c} = \vec{c} \times \vec{a} \neq \vec{0}.$$

Prove that $\vec{a} + \vec{b} + \vec{c} = \vec{0}$.

Problem 3. On the sides of triangle ABC construct in the exterior the rectangles ABB_1A_2, BCC_1B_2, CAA_1C_2. Prove that the perpendicular bisectors of A_1A_2, B_1B_2, and C_1C_2 intersect at one point.

Problem 4. Prove that if the four lines through the centroids of the four faces of a tetrahedron perpendicular to those faces are concurrent, then the four altitudes of the tetrahedron are also concurrent. Prove that the converse is also true.

Problem 5. Let $A_1, A_2, ..., A_n$ be distinct points in the plane, and let m be the number of midpoints of all the segments they determine. What is the smallest value that m can have?

Problem 6. Let M be a point in the plane of triangle ABC. Prove that the centroids of the triangles MAB, MAC, and MCB form a triangle similar to ABC.

Problem 7. Find the locus of points P in the interior of a triangle ABC such that the distances from P to the lines AB, BC and CA are the side lengths of some triangle.

Problem 8. Let $ABCDEF$ be a hexagon inscribed in a circle of radius r. Show that if $AB = CD = EF = r$, then the midpoints of BC, DE, and FA are the vertices of an equilateral triangle.

Problem 9. Let $A_1A_2...A_n$ be a regular polygon with circumradius equal to 1. Find the maximum value of $\prod_{k=1}^{n} PA_k$ as P ranges over the circumcircle.

Problem 10. Compute the integral

$$\int \frac{dx}{a + b\cos x + c\sin x}$$

where a, b, c are real numbers, not all equal to 0.

Problem 11. For a positive integer n denote by τ the permutation cycle $(n, n-1, ..., 2, 1)$. Consider the locus of points in $\mathbb{R}^n$ defined by the equation

$$\sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) x_{\sigma(1)} x_{\tau(\sigma(2))} \cdots x_{\tau^{n-1}(\sigma(n))} = 0.$$

Prove that this locus contains a plane.

Problem 12. Prove that the intersection of an n-dimensional cube centered at the origin and with edges parallel to the coordinate axes with the plane determined by the vectors

$$\vec{a} = \left(\cos\frac{2\pi}{n}, \cos\frac{4\pi}{n}, ..., \cos\frac{2n\pi}{n}\right), \quad \vec{b} = \left(\sin\frac{2\pi}{n}, \sin\frac{4\pi}{n}, ..., \sin\frac{2n\pi}{n}\right)$$

is a regular $2n$-gon.

Problem 13. Consider a unit vector starting at the origin and pointing in the direction of the tangent vector to a continuously differentiable curve in three-dimensional space. The endpoint of the vector describes the spherical image of the curve. Show that if the curve is closed, then its spherical image intersects every great circle of the sphere.

Problem 14. With the hypothesis of the previous problem, if the curve is twice differentiable, then the length of the spherical image of the curve is called the total curvature. Prove that the total curvature of a closed curve is at least 2π.

Problem 15. Prove that the plane cannot be covered by finitely many parabolas.

Problem 16. Compute the integral

$$\int \sqrt{\frac{1-x}{1+x}} dx, x \in (-1, 1).$$

Problem 17. Let n be an odd positive integer and let θ be a real number such that $\frac{\theta}{\pi}$ is irrational. Set $a_k = \tan(\theta + \frac{k\pi}{n})$, $k = 1, 2, ..., n$. Prove that

$$\frac{a_1 + a_2 + \cdots + a_n}{a_1 a_2 \cdots a_n}$$

is an integer and determine its value.

Problem 18. Find the Taylor expansion at 0 of the function

$$f(x) = e^{x\cos\theta}\cos(x\sin\theta),$$

where θ is a parameter.

Problem 19. The continuous real-valued function $\phi(t)$ is defined for $t \geqslant 0$ and is absolutely integrable on every bounded interval. Define

$$P = \int_0^\infty e^{-(t+i\phi(t))} dt, \quad Q = \int_0^\infty e^{-2(t+i\phi(t))} dt.$$

Prove that

$$|4P^2 - 2Q| \leqslant 3,$$

with equality if and only if $\phi(t)$ is constant.

Problem 20. Obtain explicit values for the following series

a. $\sum_{n=1}^\infty \arctan\frac{2}{n^2}$
b. $\sum_{n=1}^\infty \arctan\frac{8n}{n^4-2n^2+5}$.

Problem 21. Evaluate

$$\sum_{n=0}^\infty \arcsin\frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n+2}\sqrt{n+1}}.$$