COMBINATORICS AND GENERATING FUNCTIONS

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Problem 1. Let S be a nonempty set and $\mathcal{F}$ a family of $m \geqslant 2$ subsets of S. Show that among the sets of the form $A\Delta B$ with $A, B \in \mathcal{F}$ there are at least m that are distinct.

Problem 2. Find the number of permutations $a_1, a_2, a_3, a_4, a_5, a_6$ of $1, 2, 3, 4, 5, 6$ that can be transformed to the identity in exactly four transpositions and no fewer.

Problem 3. Given some positive real numbers $a_1 < a_2 < a_3 < \cdots < a_n$ find all permutations σ with the property that

$$a_1 a_{\sigma(1)} < a_2 a_{\sigma(2)} < \cdots < a_n a_{\sigma(n)}.$$

Problem 4. Determine the number of permutations $a_1, a_2, ..., a_{2004}$ of the numbers $1, 2, ..., 2004$ for which

$$|a_1 - 1| = |a_2 - 2| = \cdots = |a_{2004} - 2004| > 0.$$

Problem 5. In how many regions do n spheres divide the three-dimensional space if any two intersect along a circle, no three intersect along a circle, and no four intersect at one point?

Problem 6. In how many regions to n great circles, any three non-intersecting, divide the surface of a sphere.

Problem 7. 1981 points lie inside a cube of side length 9. Prove that there are two points within distance less than 1.

Problem 8. Inside a square of side 38 lie 100 convex polygons, each with an area at most π and perimeter at most 2π. Prove that there exists a circle of radius 1 inside the square that does not intersect any of the polygons.

Problem 9. Denote by V the number of vertices of a convex polyhedron, and by Σ the sum of the planar angles of the faces. Prove that $2\pi V - \Sigma = 4\pi$.

Problem 10. Three conflicting neighbors have three common wells. Can one draw nine paths connecting each of the neighbors to each of the wells such that no two paths intersect?

Problem 11. What is the largest number of vertices that a complete graph can have so that its edges can be colored by two colors in such a way that there is no monochromatic triangle?

Problem 12. For an arithmetic progression $a_1, a_2, ..., a_n, ...,$ let $S_n = a_1 + a_2 + ... + a_n$. Prove that

$$\sum_{k=0}^{n} \binom{n}{k} a_{k+1} = \frac{2^n}{n+1} S_{n+1}.$$

Problem 13. For a positive integer n define the integers a_n, b_n, c_n by

$$a_n + b_n 2^{\frac{1}{3}} + c_n 4^{\frac{1}{3}} = (1 + 2^{\frac{1}{3}} + 4^{\frac{1}{3}})^n.$$

Prove that

$$2^{-\frac{n}{3}}\sum_{k=0}^{n}\binom{n}{k}a_k = \begin{cases} a_n & n \equiv 0 \bmod 3 \\ b_n 2^{\frac{1}{3}} & n \equiv 2 \bmod 3 \\ c_n 4^{\frac{1}{3}} & n \equiv 1 \bmod 3 \end{cases}.$$

Problem 14. Prove that the Fibonacci sequence $F_1 = F_2 = 1$, $F_{n+1} = F_n + F_{n-1}$ satisfies

$$F_n = \binom{n}{0} + \binom{n-1}{1} + \binom{n-2}{2} + \cdots.$$

Problem 15. For a positive integer n, denote by $S(n)$ the number of choices of the signs "+" or "-" such that $\pm 1 \pm 2 ... \pm n = 0$. Prove that

$$S(n) = \frac{2^{n-1}}{\pi}\int_0^{2\pi} \cos t \cos 2t ... \cos ntdt.$$

Problem 16. Prove the identity

$$\sum_{k=1}^{n} k\binom{n}{k}^2 = n\binom{2n-1}{n-1}.$$

Problem 17. A number n of tennis players take part in a tournament in which each player plays exactly one game against each other player. Let x_i, y_i denote the number of wins and losses of player i. Show that

$$x_1^2 + x_2^2 + \cdots + x_n^2 = y_1^2 + y_2^2 + \cdots + y_n^2.$$

Problem 18. Let A be a finite set and let f and g be two functions on A. Let m be the number of pairs $(x, y) \in A \times A$ such that $f(x) = g(y)$, let n be the number of pairs (x, y) such that $f(x) = f(y)$ and let k be the number of pairs such that $g(x) = g(y)$. Prove that $2m \leqslant n + k$.

Problem 19. A sheet of paper in the shape of a square is cut by a line into two pieces. One of the pieces is cut again by a line, and so on. What is the minimum number of cuts one should perform such that among the pieces one can find one hundred polygons with twenty sides.

Problem 20. Let $S^1 = \{z \in \mathbb{C} : |z| = 1\}$. For all functions $f : S^1 \to S^1$ set $f^1 = f$ and $f^{n+1} = f \circ f^n$, $n \geqslant 1$. Call $w \in S^1$ a periodic point of f of period n if $f^i(w) \neq w$ for $i = 1, ..., n-1$ and $f^n(w) = w$. If $f(z) = z^m$, m a positive integer, find the number of periodic points of f of period 1989.