PUTNAM SEMINAR

MULTIVARIABLE ANALYSIS.

(PUTNAM THIS YEAR WILL BE UNOFFICIAL)

LAGRANGE MULTIPLIERS:

CONSTRAINED OPTIMIZATION

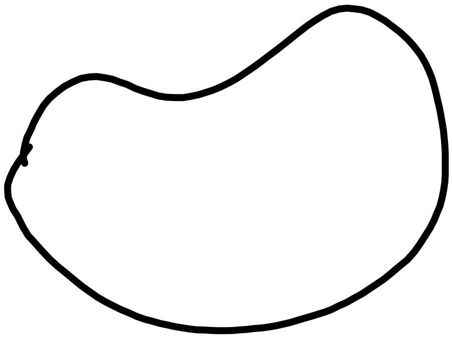
MAXIMIZE OR MINIMIZE

$f(\underline{x})$ SUBJECT TO A CONSTRAINT $G(\underline{x})=C$.

IF G IS A REAL VALUED FN, THE OPTIMUM OCCURS WHERE

∇f IS PARALLEL TO ∇G.

IF THERE ARE SEVERAL CONSTRAINTS ∇f IS IN THE SPAN OF $\nabla G_1, \ldots, \nabla G_k$.



$\gamma(t)$ IS A PATH ON THE CURVE.

$G(x) = c$. $G(\gamma(t)) = c$.

$$\frac{d}{dt} G(\gamma(t)) = \nabla G \cdot \gamma'(t) = 0.$$

AT A MAX/MIN

$\frac{d}{dt} f(\gamma(t)) = 0$, ALSO,

$$\nabla f(\gamma(t)) \cdot \gamma'(t) = 0.$$

THIS HOLDS FOR ALL CURVES THROUGH THE POINT.

[∇f IS $\perp$ TO THE TANGENT SPACE OF THE CONSTRAINT CURVE].

CAUCHY'S THEOREM:

IF f IS ANALYTIC ON A BOUNDED REGION Δ, $a \in \Delta$, Γ BOUNDING CURVE,

$$\frac{1}{2\pi i}\int_{\Gamma} \frac{f(z)}{(z-a)^{n+1}}\,dz = \frac{f^{(n)}(a)}{n!}.$$

A TYPICAL USE CHOOSES Γ IN A WAY TO OPTIMIZE THE OUTCOME. [SEE PUTNAM B6, 2018 (?))

THIS IS USED FREQUENTLY WITH f A GENERATING FUNCTION.

STOKES' THEOREM:

M n DIMENSIONAL MANIFOLD (ASSUME EMBEDDED IN $\mathbb{R}^k$ SOME k FOR OUR PURPOSES)

∂M. ASSUME M, AND ∂M ARE ORIENTABLE.

LET ω BE A DIFFERENTIAL $n-1$ FORM.

THEN $\int_{\partial M} \omega = \int_M d\omega$

d IS THE EXTERIOR DERIVATIVE.

$$\int_B f(x)\, dx_1 \wedge \dots \wedge dx_n$$

$B \subset \mathbb{R}^n$ IS JUST THE VOLUME INTEGRAL OF f.

$$= \int_B f\, dV.$$

$$df(x) = \frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2 + \dots + \frac{\partial f}{\partial x_n} dx_n.$$

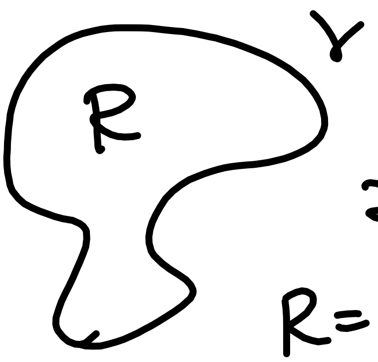
$$d\left[f(x) \wedge dx_{i_1} \wedge \dots \wedge dx_{i_j}\right)$$

$$(df) \wedge dx_{i_1} \wedge \dots \wedge dx_{i_j}.$$

THE WEDGE PRODUCT IS CALCULATED BY SORTING $x_{i_1}, \dots, x_{i_j}$ INTO

SORTED ORDER, CHANGING THE SIGN EACH TIME TWO INDICES ARE SWAPPED.

GREEN'S THEOREM:



$$\int_{\partial M=\gamma} \underbrace{F\,dx + G\,dy}_{\omega}.$$

$R=M$

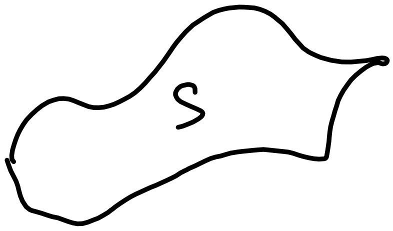
$$d\omega = \cancel{\frac{\partial F}{\partial x}\, dx\wedge dx}^{\,0} + \frac{\partial F}{\partial y}\, \overbrace{dy\wedge dx}^{-dx\wedge dy}$$

$$+ \frac{\partial G}{\partial x}\, dx\wedge dy + \cancel{\frac{\partial G}{\partial y}\, dy\wedge dy}^{\,0}$$

$$= \iint_R \frac{\partial G}{\partial x} - \frac{\partial F}{\partial y}\, dx\wedge dy.$$

THIS IS THE USUAL STATEMENT OF GREEN'S THEOREM.

CLASSICAL STOKES THEOREM:



$S \subset \mathbb{R}^3$

$\partial S = \gamma$.

WE WISH TO CALCULATE THE LINE INTEGRAL $\oint_\gamma F \cdot d\vec{R}$

WHERE F IS A VECTOR FIELD.

THIS IS EQUAL TO

$$\iint_S \operatorname{curl} F \cdot N \, dS.$$

NORMAL VECTOR

GIVEN THE 1-FORM

$$F = F_1 dx + F_2 dy + F_3 dz.$$

$$dF = \frac{\partial F_1}{\partial y} dy \wedge dx + \frac{\partial F_1}{\partial z} dz \wedge dx + \frac{\partial F_2}{\partial x} dx \wedge dy + \frac{\partial F_2}{\partial z} dz \wedge dy + \frac{\partial F_3}{\partial x} dx \wedge dz + \frac{\partial F_3}{\partial y} dy \wedge dz$$

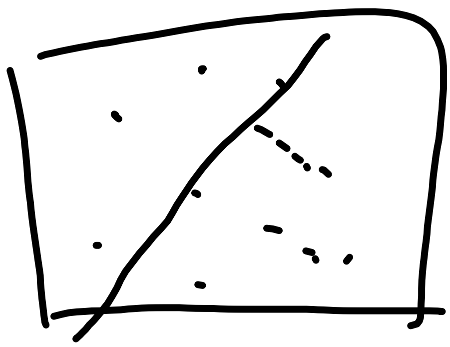
$$= \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx \wedge dy + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) dz \wedge dx + \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) dy \wedge dz.$$

THE COORDINATES OF THIS ARE THE SAME AS THE CURL, WHICH GIVES THE CLAIMED FORMULA.

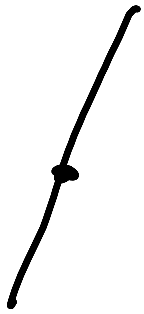
PROBLEMS TO PRESENT:

3 4 15

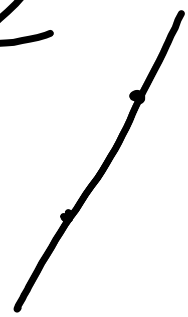
3



n=1



n=2



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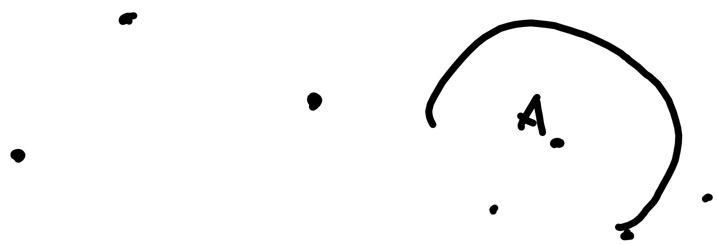
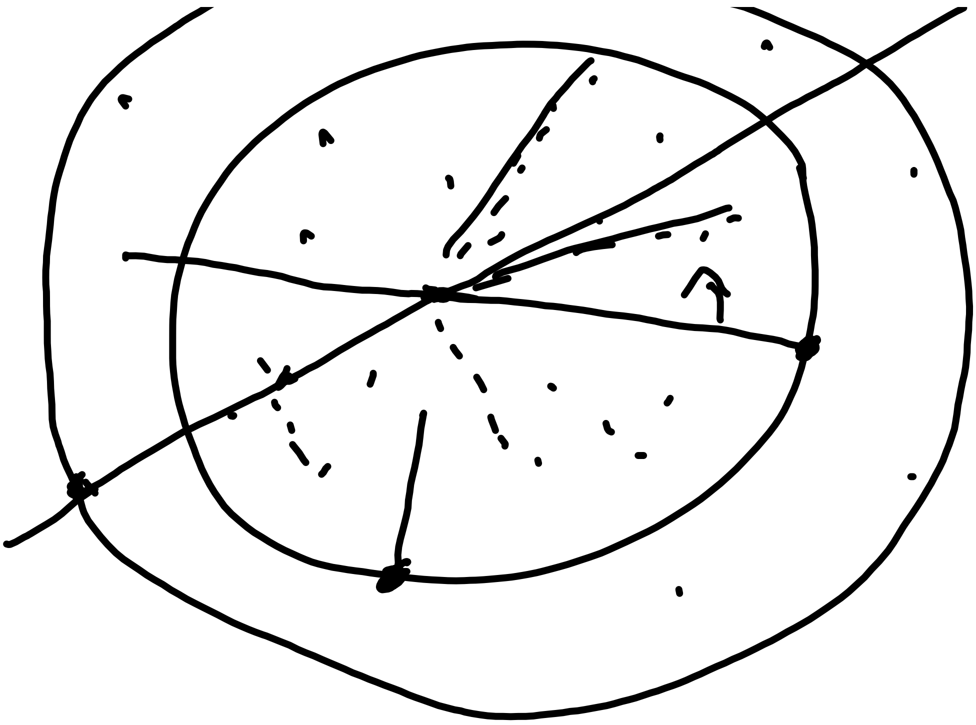
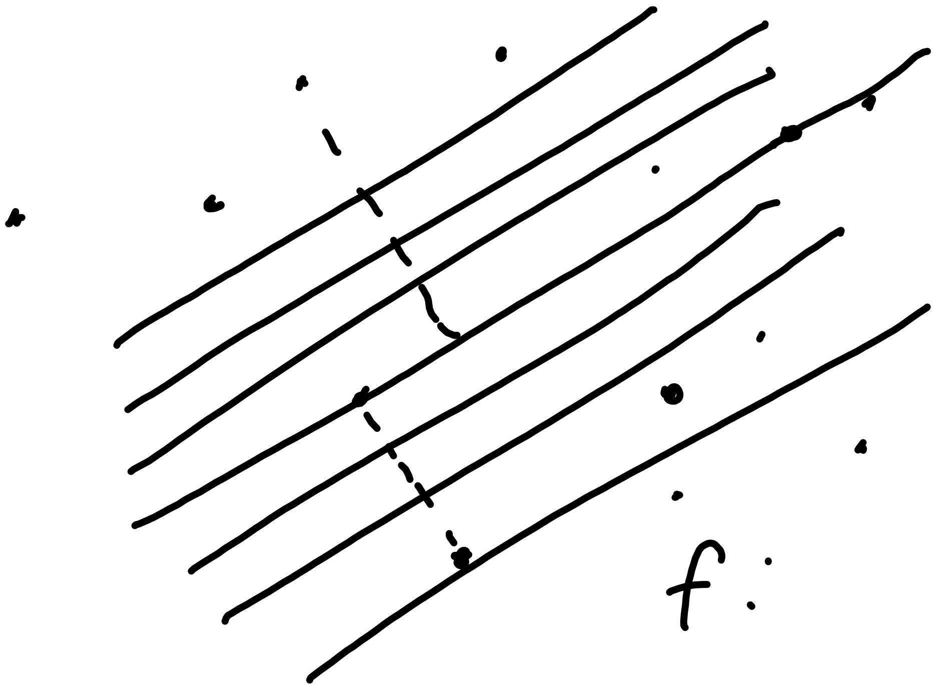
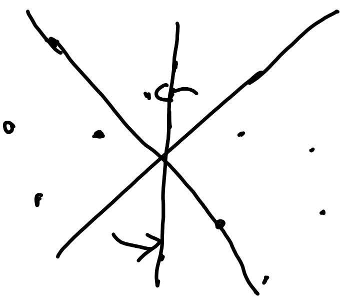
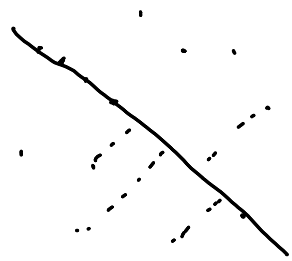


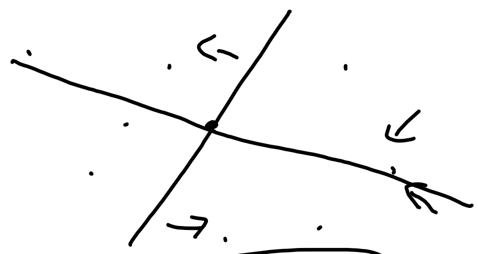


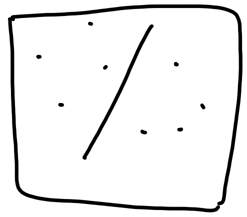


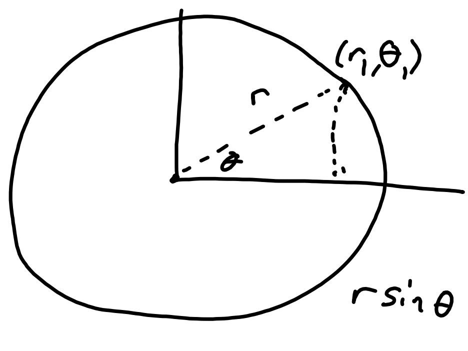
f :



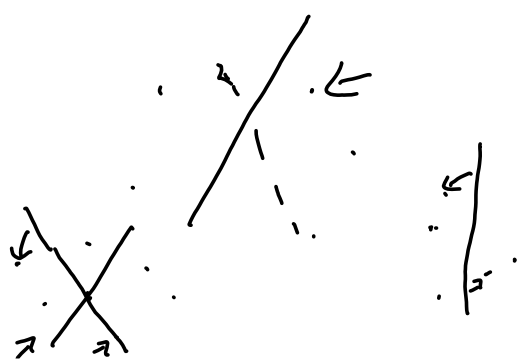


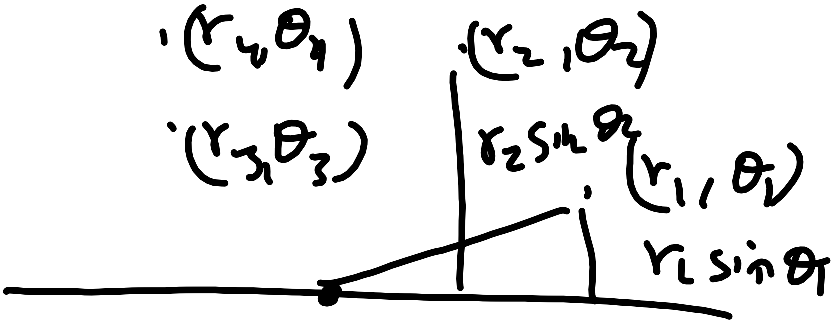






(r1,θ1)
r
θ
r sin θ





SUM OF DISTANCES:
$\sum r_i |\sin\theta_i|$.

ROTATE BY θ:

$$f(\theta) = \sum r_i |\sin \theta_i - \theta|.$$

ASSUME AT A MIN NO

$\sin\theta_i - \theta = 0$

THEN f DIFF,

$$f'(\theta) = 0 = \sum r_i \operatorname{sign}(\sin\theta_i - \theta) \cos\theta_i - \theta.$$

$$f''(\theta) = \sum r_i \operatorname{sgn}(\sin\theta_i - \theta)(-\sin\theta_i - \theta)$$

$$= -f(\theta). < 0.$$

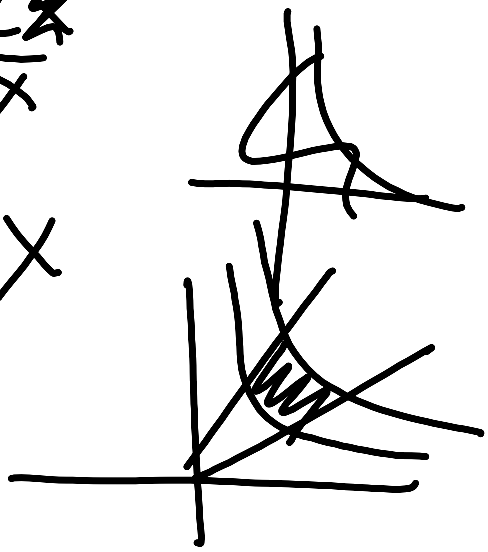
CONTRADICTS MINIMUM.
THAN MIN OCCURS WITH SOME $\sin(\theta_i - \theta) = 0$

4 $\iint_D x \, dx \, dy$

$$D = \{ x \geq 0, \ 1 \leq xy \leq 2, \ 1 \leq \frac{y}{x} \leq 2 \}$$

$$\frac{1}{x} \le y \le \frac{2\cancel{x}}{x}$$

$$x \le y \le 2x$$



$u = xy \qquad x = \sqrt{\frac{u}{v}}$

$v = \frac{y}{x} \qquad y = \sqrt{uv}$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix} = \frac{1}{4}\left(\frac{1}{v} + \frac{1}{\sqrt{v}}\right)$$

$$\iint_D x\,dx\,dy = \int_1^2 \int_1^2 \sqrt{\frac{u}{v}}\left(\frac{1}{4}\left(\frac{1}{v}+\frac{1}{\sqrt{v}}\right)\right)$$

$u = xy.$

$u' = y + xy'$

$u'' = y' + y' + xy''.$

$= 2y' + xy''!$

$u'' + u = 0.$ So u is lin combo. of sin, cos,

(9)

WISH TO CONSIDER

$$\iiint_B f\Delta g - g\Delta f.\ dV$$

WE WISH TO WRITE THIS

AS $\iiint_B \operatorname{div} F \ dV$

$$= \iint_{S^2} F\cdot N \ dS.$$

$F = f\nabla g - g\nabla f.$

$$\operatorname{div} F = \cancel{\nabla f\cdot\nabla g} + f\cdot\Delta g$$
$$- \cancel{\nabla g\cdot\nabla f} - g\cdot\Delta f.$$
$$= f\Delta g - g\Delta f.$$

$$\iint_S (f\nabla g - g\nabla f)\cdot N \ dS.$$

BUT f AND g ARE CONSTANT IN THE RADIAL DIRECTION,

SO $\nabla f\cdot N = \nabla g\cdot N = 0.$