# PUTNAM SEMINAR

POLYNOMIALS.

USEFUL FACTS ABOUT POLYNOMIALS

POLYNOMIAL IN SEVERAL VARIABLES

$p(x_1, \ldots, x_n)$ THERE IS GROUP ACTION OF THE SYMMETRIC GROUP $S_n$,

$$\sigma \cdot P(x_1, \ldots, x_n) = P(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$$

FOR INSTANCE: IF $\sigma = (1,2)$

$p(x) = x_1^2 + x_2^3 + x_3$

$\sigma \cdot P(x) = x_2^2 + x_1^3 + x_3.$

A POLYNOMIAL WHICH IS FIXED BY THE SYMMETRIC GROUP ACTION IS A SYMMETRIC POLYNOMIAL.

EXAMPLES OF SYMMETRIC POLYNOMIALS

ELEMENTARY SYMMETRIC POLYS:
THESE HAVE DEGREE 1 IN EACH VARIABLE.

$e_1 = x_1 + x_2 + \cdots + x_n$

$e_2 = x_1x_2 + x_1x_3 + \cdots + x_{n-1}x_n$

$e_3 = x_1x_2x_3 + x_1x_2x_4 + \cdots + x_{n-2}x_{n-1}x_n.$

$\vdots$

$e_n = x_1x_2x_3\cdots x_n.$

HOMOGENEOUS SYMMETRIC POLYS:

THESE ALLOW HIGHER POWERS:

$$h_1 = e_1 = x_1 + x_2 + \cdots + x_n$$

$$h_{21} = x_1^2 x_2 + x_1^2 x_3 + \cdots + x_1 x_2^2 + x_1 x_3^2 + \cdots + x_{n-1} x_n^2 .$$

$$h_{321} = x_1^3 x_2^2 x_3 + x_1^3 x_2^2 x_4 + \cdots + x_{n-2} x_{n-1}^2 x_n^3$$

POWER SUM SYMMETRIC POLYS:

$$p_1 = x_1 + x_2 + \dots + x_n$$

$$p_2 = x_1^2 + x_2^2 + \dots + x_n^2$$

$$p_3 = x_1^3 + x_2^3 + \dots + x_n^3$$

$$\vdots$$

ALL OF THESE ARE ALGEBRA BASES FOR THE RING OF SYMMETRIC POLYNOMIALS.

GIVEN A POLY $P(x)$

FORM $\mathrm{Sym}(P)(\underline{x}) = \frac{1}{n!} \sum_{\sigma \in S_n} \sigma \cdot P(\underline{x})$

THIS IS NOW A SYMMETRIC POLY

SINCE IF $\tau$ IS A PERM.

$\tau \cdot \mathrm{Sym}(P)(\underline{x}) = \frac{1}{n!} \sum_{\sigma \in S_n} \tau \cdot (\sigma \cdot P(x))$

$= \mathrm{Sym}(P)(\underline{x})$.

THIS STILL RUNS OVER ALL PERMS ONCE.

IF $P$ WAS SYMMETRIC, $\mathrm{Sym}(P) = P$.

FUNDAMENTAL THM OF ALGEBRA:

ANY POLYNOMIAL $P(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_0$

FACTORS INTO LINEAR FACTORS OVER $\mathbb{C}$:

$a_n (x-x_1)(x-x_2)\cdots(x-x_n)$.

THE VIETE RELATIONS RELATE THE COEFFICIENTS TO THE ELEMENTARY SYM FNS IN THE ROOTS.

$$x_1 + x_2 + \dots + x_n = -\frac{a_{n-1}}{a_n}$$

$$x_1x_2 + x_1x_3 + \dots + x_{n-1}x_n = \frac{a_{n-2}}{a_n}$$

$$\vdots$$

$$x_1 \cdots x_n = (-1)^n \frac{a_0}{a_n}.$$

EVIDENTLY, IF YOU SYMMETRIZE A POLYNOMIAL BY AVERAGING, THE RESULT IS A LINEAR COMB. OF HOMOGENEOUS SYMMETRIC POLYNOMIALS.

IN FACT, PRODUCTS OF POWER SUM SYMMETRIC POLYS SPAN THE HOMOGENEOUS ONES:

EG:

$$h_{31} = x_1^3 x_2 + x_1^3 x_3 + \dots + x_1^3 x_n + \dots + x_{n-1} x_n^3$$

$$h_{31} - p_3 \cdot p_1 = \dots$$

$$p_3 p_1 = (x_1^3 + \dots + x_n^3)(x_1 + \dots + x_n)$$
$$= h_{31} + x_1^4 + x_2^4 + \dots + x_n^4.$$

So $h_{31} = p_3 p_1 - p_4$.

PRODUCTS OF ELEMENTARY SYM FNS ALSO SPAN.

GAUSS LUCAS THM:

P(Z) COMPLEX POLY. THE ROOTS OF P'(Z) ARE IN THE CONVEX HULL OF THE ROOTS OF P(Z).

EISENSTEIN CRITERION:

$P(x) \in \mathbb{Z}[x]$. $P(x) = a_n x^n + \dots + a_0$, $p$ PRIME, $p \nmid a_n$, $p \mid a_{n-1}, \dots, a_0$ $p^2 \nmid a_0$ THEN $P(x)$ IS IRREDUCIBLE.

PROBLEMS TO PRESENT:

~~1~~ ~~10~~ ~~2~~ 12 14

(1) $x = \sqrt{2} + \sqrt{3}$

$$(x - \sqrt{2})^2 = 3$$

$$x^2 - 2\sqrt{2}x + 2 = 3$$

$$x^2 - 1 = 2\sqrt{2}x$$

$(x^2-1)^2 = 8x$

$x^4+1-2x^2 = 8x^2$

$x^4-2x^2-8x^2+1$

$x^4-10x^2+1$

(10) Suppose [illegible] $(x)$

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Note $p(x) \geq 1 \Rightarrow q(x)\ p(x)$ something

WLOG, we assume $q(x), p(x) > 0$

$p(a_i) = 1 \Rightarrow q(a_i) = p(a_i) = 1$

Let $f(x) = q(x) - 1$ $g(x) = p(x) - 1$

$a_i$ is the root of $f(x), g(x)$

Next. sue degree of $P(x)$ $n^2$

$f(x), g(x)$ $n$ $a_i$

$f(x) = A\underline{(x-a_1)(x-a_2)\cdots(x-a_n)}$

$g(x) = B\underline{(x-a_1)(x-a_2)\cdots(x-a_n)}$

$p(x) = \underline{P f(x)+1}$ $\quad q(x) = C\underline{(g(x)+1}$

$P(x) = p(x)q(x) = \underline{(Df(x)+1)(Cg(x)+1}$

$$\begin{cases} C + D = 1 \\ CD = 1 \end{cases} \qquad p(a_i) = 1$$

$C, D$

$$(Df(x)+1)(Cg(x)+1) = CD\,f(x)g(x) + 1 + Df(x) + Cg(x)$$

(2) $P(x) = x^4 + ax^3 + bx^2 + cx + d$

$Q(x) = x^2 + px + q$

BOTH NEG. ON $(r, s)$, POS. FOR $x < r$ OR $x > s$.

CONCLUDE: $Q(x)$ AND $P(x)$ HAVE ROOTS AT $r, s$. THUS $Q(x) \mid P(x)$. $Q(x) = (x-r)(x-s)$.

$P(x)$ HAS ROOTS AT $r, s$ ALSO.

$R(x) = \dfrac{P(x)}{Q(x)}$ WHICH IS A POLY, NON-NEGATIVE, DEGREE 2, MONIC.

$R(x) = x^2 + ux + v$.

NON-NEG. $\Rightarrow$ $(x - \frac{u}{2})^2 + v - \frac{u^2}{4}$.

$v \geq \frac{u^2}{4}$.

$(x - \frac{u}{2})^2 + v - \frac{u^2}{4} \geq (x - \frac{u}{2})^2$.

OVER AN INTERVAL OF LENGTH $> 2$, THE MAX OF THIS POLY IS $> 1$, THUS BETWEEN $r, s$ $\exists\, x$ S.T. $R(x) > 1$.

$\Rightarrow P(x) < Q(x)$.

$s - u/2$ or

$(r \quad u/2 \quad s)$ $\square$

(1c) FIND $F(x), G(x)$ SO THAT $(x^8-1)F(x)+(x^5-1)G(x)=x-1$.

EUCLIDEAN ALGORITHM:

$(x-1)(x^7+x^6+\dots+1)F(x)+(x-1)(x^4+x^3+\dots+1)G(x)$

$=(x-1)$

SO WE WANT

$f(x)(x^7+x^6+\dots+1)+G(x)(x^4+x^3+\dots+1)=1$

$(x^4+x^3+\dots+1)\overline{)x^7+x^6+\dots+1}^{\,x^3}$ R $x^2+x+1$

$(x^7+x^6+\dots+1)=(x^2+x+1)+x^3(x^4+\dots+1)$

$(x^4+x^3+\dots+1)=x^2(x^2+x+1)+x+1$

$(x^2+x+1)=(x+1)\cdot x+1$

$1=(x^2+x+1)-x(x+1)$

$(x+1)=(x^4+x^3+\dots+1)-x^2(x^2+x+1)$

$1=(x^2+x+1)-x(x^4+x^3+\dots+1-x^2(x^2+x+1))$

$=(1-x^3)(x^2+x+1)-x(x^4+x^3+\dots+1)$

$x^2+x+1=x^7+\dots+1-x^3(x^4+x^3+\dots+1)$

SUBSTITUTE ABOVE.