PUTNAM SEMINAR
WEEK 3

INEQUALITIES.

CAUCHY - SCHWARZ
INEQUALITY.

IN AN INNER PRODUCT SPACE, SO $\langle , \rangle$ IS A BILINEAR PAIRING,

$$|\langle v, w \rangle| \leq \|v\| \cdot \|w\|$$

$$\|v\| = \langle v, v \rangle^{1/2}.$$

<u>FOR EXAMPLE:</u> IN $\mathbb{R}^n$,

$$\langle \underline{x}, \underline{y} \rangle = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$$

IN THIS CASE THE INEQUALITY SAYS

$$|x_1y_1+x_2y_2+\cdots+x_ny_n| \leq (x_1^2+\cdots+x_n^2)^{1/2}(y_1^2+\cdots+y_n^2)^{1/2}.$$

IN GENERAL A BILINEAR PAIRING $\langle\, , \rangle \to \mathbb{R}$ SATISFIES:

(1) $\langle x, x\rangle \geq 0$, EQUALITY IFF $x = 0$

(2) LINEAR IN EACH VARIABLE

$$\langle ax+by, z\rangle = a\langle x, z\rangle + b\langle y, z\rangle$$

(3) SYMMETRIC $\langle x, y\rangle = \langle y, x\rangle$.

TO PROVE CAUCHY-SCHWARZ

$0 \leq \langle x - ty, x - ty \rangle$ ALL $t \in \mathbb{R}$.

$= \langle x,x \rangle - 2t\langle x,y \rangle + t^2\langle y,y \rangle$.

IF EITHER $x, y = 0$, EQUALITY.
OTHERWISE, CHOOSE t TO MINIMIZE THE QUADRATIC EQ'N

MINIMUM OF $At^2 + Bt + C$
OCCURS AT $t = \frac{-B}{2A}$.

AND IS EQUAL TO
$A\left(t - \frac{B}{2A}\right)^2 + \left(C - \frac{B^2}{4A}\right)$

MINIMUM IS $C - \frac{B^2}{4A}$.

THIS IMPLIES
$0 \leq \langle x,x \rangle - \frac{4\langle x,y \rangle^2}{4\langle y,y \rangle}$

THIS IMPIES
$\langle x,y \rangle^2 \leq \langle x,x \rangle\langle y,y \rangle$. $\square$

EQUALITY HOLDS IFF
X AND Y ARE LINEARLY INDEPENDENT.

EXAMPLE: IF $|x|<1$, $\sum_{k=0}^{\infty} a_k^2 < \infty$ THEN

$$\left|\sum_{k=0}^{\infty} a_k x^k\right| \leq \frac{1}{\sqrt{1-x^2}}\left(\sum_k a_k^2\right)^{1/2}$$

PROOF: $\ell^2 = \{(a_k)_{k=0}^{\infty} : \sum a_k^2 < \infty\}$.

THIS HAS A BILINEAR PAIRING

$$\langle (a_k)_k, (b_k)_k \rangle = \sum_{k=0}^{\infty} a_k b_k.$$

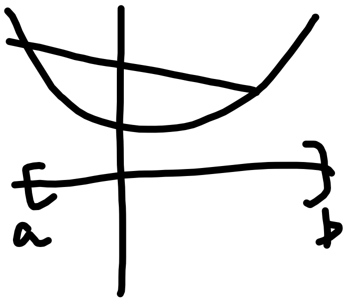
$$\left|\sum_{k=0}^{\infty} a_k x^k\right| = \left|\langle (a_k), (x^k) \rangle\right|$$
$$\leq \|a_k\|_2 \cdot \|(x^k)\|_2.$$
$$= \left(\sum a_k^2\right)^{1/2} \left(\sum_{k=0}^{\infty} x^{2k}\right)^{1/2}$$
$$\left(\frac{1}{1-x^2}\right)^{1/2}. \quad \square$$

JENSEN'S INEQUALITY:

$f:[a,b] \to \mathbb{R}$ BE CONVEX

$\forall x,y \in [a,b]$,

$$f\left(\frac{x+y}{2}\right) \le \frac{f(x)+f(y)}{2}$$



IF $0 \le p_1, \dots, p_n$,

$\sum p_j = 1$

THEN $\sum p_j f(x_j) \ge f\left(\sum p_j x_j\right)$

INTEGRAL VERSION:

IF μ IS A PROBABILITY MEASURE ON $\mathbb{R}$, THE

$\mu(A) \geq 0$, $\mu(\mathbb{R}) = 1$.

THEN $\int f(x)\, d\mu(x) \geq f\left(\int x\, d\mu(x)\right)$

EXAMPLES OF PROBABILITY MEASURES

- dx ON $[0,1]$. PICK A UNIFORM POINT ON $[0,1]$
- $\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$ GAUSSIAN DENSITY ON $\mathbb{R}$.
- e^{-x} ON $[0,\infty)$ EXPONENTIAL DISTRIBUTION

ASSIGN WEIGHT
p_j TO x_j
IS AN EXAMPLE OF
A PROBABILITY MEASURE,
SINCE $p_j \geq 0$, $\sum p_j = 1$.

HÖLDER'S INEQUALITY:

DEFINE THE ℓ^p NORM

$$\|\underline{x}\|_p = \left(x_1^p + x_2^p + \dots + x_n^p\right)^{1/p}. \quad p \geq 1.$$

$q = \frac{p}{p-1}$, $\frac{1}{p} + \frac{1}{q}$ IS A CONJUGATE EXPONENT.

$$|\underline{x} \cdot \underline{y}| \leq \|\underline{x}\|_p \cdot \|\underline{y}\|_q.$$

INTEGRAL VERSION $\left|\int f(x) g(x) \, d\mu(x)\right|$

$$\leq \left(\int |f(x)|^p \, d\mu(x)\right)^{1/p} \left(\int |g(x)|^q \, d\mu(x)\right)^{1/q}$$

μ PROB. MEASURE.

PROOF OF HÖLDER'S INEQUALITY:

WE CAN ASSUME THAT EACH $x_j \neq 0$, OR DISCARD THE INDEX j.

LET $w_j = \dfrac{|x_j|^p}{\sum_k |x_k|^p}$, SO $w_1+\dots+w_n=1$.

WE CAN CHECK, BY JENSEN,

$$\left(\sum_j \left(\frac{|y_j|}{|x_j|^{p-1}}\right) w_j\right)^{\frac{p}{p-1}} \le \sum_j \left(\frac{|y_j|}{|x_j|^{p-1}}\right)^{\frac{p}{p-1}} w_j$$

$$f(\textstyle\sum x_j p_j) \qquad \sum_j f(x_j) p_j$$

BECAUSE THE FUNCTION $x \mapsto x^{\frac{p}{p-1}}$ IS CONVEX.

LHS: $\dfrac{w_j}{|x_j|^{p-1}} = \dfrac{|x_j|^p}{|x_j|^{p-1}(\sum |x_k|^p)} = \dfrac{|x_j|}{\sum |x_k|^p}$.

SO LHS: $\left(\dfrac{\sum_j |x_j||y_j|}{\sum |x_k|^p}\right)^{\frac{p}{p-1}}$

RHS: $\sum \left(\dfrac{|y_j|}{\cancel{|x_j|^{p-1}}}\right)^{\frac{p}{p-1}} \dfrac{\cancel{|x_j|^p}}{\sum |x_k|^p}$

$= \sum |y_j|^{p/p-1} / \sum |x_k|^p$

THUS WE'VE FOUND:

$$\left(\sum |x_j||y_j|\right)^{p/p-1} \le \sum |y_j|^{\frac{p}{p-1}} \left(\sum |x_j|^p\right)^{\frac{p}{p-1}-1}$$

SO $\sum |x_j||y_j| \le \left(\sum |y_j|^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}} \left(\sum |x_j|^p\right)^{\frac{1}{p}}$. $\square$

PROBLEMS TO PRESENT:

4 2 5 9 8

There are at least 2 ways
to make

$$\pm x^{n+2} \pm x^{n+1} \leq 0$$

$x > 0$: $\boxed{-x^2 - x} < 0$

$x > 1 \Rightarrow \boxed{x - x^2} < 0$

$x = 1 \Rightarrow \boxed{x^2 - x} = 0.$

$x^2 < x \Rightarrow \boxed{x^2 - x} < 0$

$x < 0$: $\boxed{-x^2 + x} < 0$

$\boxed{-x - x^2} < 0$ if $x > -1$

$x = -1$ $\boxed{x^2 - x} = 0$

$x < -1$ $\boxed{x^2 + x} < 0$

$$\pm x^{n+2} \pm x^{n+1}$$

$k = 2.$

$$\pm x^2 \pm x^1 \leq 0 < \frac{1}{2}$$

$$2 = 2^{\frac{2}{1}} \text{ ways}$$

$k = n$, $\pm x^n \pm x^{n-1} \pm \cdots \pm x$

$2^{\frac{n}{2}}$ ways.

$n+2$

$$\pm x^{n+2} \pm x^{n+1} \pm x^n \pm \cdots x$$

≤ 0 $\quad < \frac{1}{2}$

2 ways $\quad 2^{\frac{n}{2}}$ ways $\quad 2 \times 2^{\frac{n}{2}} = 2^{\frac{n+2}{2}}$

(2) $2^x + 3^x - 4^x + 6^x - 9^x \leq 1$

$\Leftrightarrow (2^x - 1)^2 + (3^x - 1)^2 - (1 - 2^x)(1 - 3^x) \geq 0$

$\Leftrightarrow \quad a^2 + b^2 - ab \geq 0$

$a = 2^x - 1 \quad b = 3^x - 1$

$ab > 0$

(3) $a_1 + \cdots a_n = 1$

prove $a_1^4 + \cdots a_n^4 \geq n$

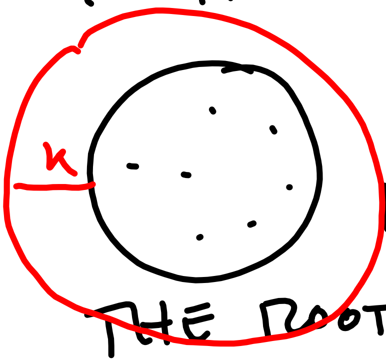
Let $f(x) = x^4$ let $P_1 \cdots P_n = \frac{1}{n}$

$$f\left(\sum \frac{1}{n} a_i\right) \leq \sum \frac{1}{n} f(x_i)$$

$$\underbrace{\frac{1}{n^3} f\left(\sum a_i\right)}_{n^4} \leq \sum a_i^4$$

$$n \leq \sum a_i^4$$

(8) P, A POLYNOMIAL OF DEGREE n. ROOTS CONTAINED IN A DISC OF RADIUS R.



$$Q(x) = nP(x) - kP'(x).$$

WE WANT SHOW THE ROOTS ARE CONTAINED IN A DISC OF RADIUS $R+k$.

$$\frac{Q(x)}{nP(x)} = 1 - \frac{k}{n}\frac{P'(x)}{P(x)}. \qquad \frac{P'}{P}(x) = \frac{d}{dx}\log P(x)$$

$P(x) = A(x-x_1)\cdots(x-x_n)$ THEN

$$\frac{P'(x)}{P(x)} = \frac{1}{x-x_1} + \frac{1}{x-x_2} + \dots + \frac{1}{x-x_n}.$$

IF x IS OUTSIDE A DISC OF RADIUS $R+k$, THEN $|x-x_j| \geq k$.

THUS

$$\frac{Q(x)}{nP(x)} = 1 - \frac{k}{n}\sum_{j=1}^{n}\frac{1}{x-x_j}$$

EACH TERM $\leq \frac{1}{n}$ IN SIZE.

$$|\quad| < 1.$$

□

9. $x_1 x_2 x_3 \dots x_n - y_1 y_2 \dots y_n$

$$x_1 \dots x_n - x_1 \dots x_{n-1} y_n$$
$$+ x_1 \dots x_{n-1} y_n - x_1 \dots x_{n-2} y_{n-1} y_n$$
$$+ \vdots$$
$$+ x_1 y_2 \dots y_n - y_1 \dots y_n$$

$$\| V_1 \dots V_n x - W_1 \dots W_n x \|$$

$$\leq \| V_1 \dots V_n x - V_1 \dots V_{n-1} W_n x \|$$
$$+ \| V_1 \dots V_{n-1} W_n x - V_1 \dots V_{n-2} W_{n-1} W_n x \|$$
$$+ \vdots$$
$$+ \| V_1 W_2 \dots W_n x - W_1 \dots W_n x \|$$

V_i, W_i ARE ISOMETRIES, COMMON ISOMS MAY BE ELLIMINATED.

$$\| V_1 \dots V_{k-1} V_k W_{k+1} \dots W_n x - V_1 \dots V_{k-1} W_k \dots W_n x \|$$
$$= \| V_k \underbrace{W_{k+1} \dots W_n x} - W_k \underbrace{W_{k+1} \dots W_n x}_{y} \|$$
$$= \| V_k y - W_k y \| \leq 1$$

SINCE $\|y\| \leq 1$. $\square$

COMMENTS:

(10) IS CALLED THE BIRTHDAY PROBLEM.

Q^n: IF 25 PEOPLE ARE IN A ROOM, WHAT IS THE PROBABILITY THAT HAVE THE SAME BIRTHDAY?

ANSWER: $1 - \frac{365}{365} \times \frac{364}{365} \times \frac{363}{365} \cdots \frac{341}{365}$.

$\boxed{> \frac{1}{2}}$

TRY ESTIMATING THE PRODUCT BY TAYLOR EXPANSION.

(19) HEISENBERG'S INEQUALITY.

A MATHEMATICAL FORMALISM OF THE HEISENBERG UNCERTAINTY PRINCIPLE.

IT SAYS THAT A FUNCTION AND ITS FOURIER TRANSFORM CANNOT BE LOCALIZED AT THE SAME TIME.

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{2\pi i \xi \cdot x} \, dx$$

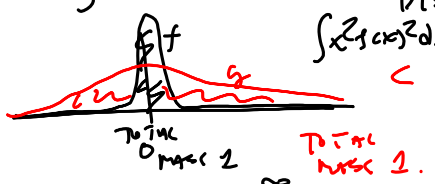
Beware missing constants here.

$$\frac{d}{d\xi}\hat{f}(\xi) = \int_{-\infty}^{\infty} x f(x) e^{2\pi i \xi \cdot x} \, dx$$

$$\left\| \frac{d}{d\xi} \hat{f}(\xi) \right\|_2^2 = \underbrace{\int_{-\infty}^{\infty} x^2 f(x)^2 dx}.$$

2ND MOMENT, A MEASURE OF THE SPREAD OF THE MASS OF $f(x)^2$.

$\int f(x)^2 dx = 1$ ⟵ MASS DISTRIBUTION.



$\int x^2 f(x)^2 dx < \int x^2 g(x)^2 dx$

$$1 = \int_{-\infty}^{\infty} f(x)^2 dx = c(\ldots)\int_{-\infty}^{\infty} |\hat{f}(\xi)|^2 d\xi$$

$$\le C(\ldots)\left(\int_{-\infty}^{\infty} x^2 f(x)^2 dx\right)^{1/2}\left(\int_{-\infty}^{\infty} \xi^2 |\hat{f}(\xi)|^2 d\xi\right)^{1/2}$$