2020 PUTNAM SEMINAR

MEETS 6-8 PM FRIDAYS.

EXAM: FEB 20, 2021

math.stonybrook.edu/~rdhough/putnam-seminar/2020/2020.html

ROBERT HOUGH

INDUCTION, PIGEONHOLE, AND EXTREMAL CONFIGURATIONS.

INDUCTION:

TO PROVE A STATEMENT $P(k)$ FOR EACH NATURAL NUMBER k

(1) BASE CASE: PROVE $P(1)$

(2) INDUCTIVE STEP: PROVE $P(k) \Rightarrow P(k+1)$

THEOREM: LET THE FIBONACCI SEQUENCE BE GIVEN BY

$F_0 = F_1 = 1, \quad F_{k+1} = F_k + F_{k-1}$

$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots$

EVERY NATURAL NUMBER $n \geq 1$ CAN BE WRITTEN AS A SUM OF DISTINCT FIBONACCI NUMBERS

$n = F_{k_1} + F_{k_2} + \cdots + F_{k_r}$

$k_1 > k_2 > k_3 > \cdots$ AND

$k_i \geq k_{i-1} + 2.$

EXAMPLE: $7 = 5 + 2$

$F_0 = 1, F_1 = 1, F_2 = 2, F_3 = 3, F_4 = 5$

$7 = F_4 + F_2.$

PROOF: BY INDUCTION.

(1) BASE CASE: $1 = F_1$ IS A FIBONACCI NUMBER.

(2) INDUCTIVE STEP: SUPPOSE FOR SOME $n \geq 1$,

$n = F_{k_1} + \cdots + F_{k_r}$ $\quad k_i \geq k_{i+1} + 2$ ALL i.

THEN

$n+1 = F_{k_1} + \dots + F_{k_r} + F_1$ 1.

IF $k_r \geq 3$ WE'RE DONE.

LEMMA: SUPPOSE $(k_{s+1} > k_s)$ ONLY LAST TWO ~~DIFFER~~ BY ≤1.

$m = F_{k_1} + F_{k_2} + \dots + F_{k_s} + F_{k_{s+1}}$

WHERE $k_i \geq k_{i+1} + 2$ IF $i \in 1, 2, \dots, s-1$

THEN m CAN BE REPRESENTED AS A SUM OF FIBONACCI NUMBERS WITH INDICES DIFFERING BY ≥2.

THE LEMMA SUFFICES TO PROVE THM.

PROOF OF LEMMA:

BY INDUCTION ON S:

(1) BASE CASE: IF $S = 1$

$F_{k_s} + F_{k_{s+1}} = F_{k_s - 1}$ IF $k_{s+1} = k_s + 1$, AND OTHERWISE ALREADY IS OF THE DESIRED TYPE.

(2) STRONG INDUCTIVE STEP: ASSUME TRUE IF $S \leq n$.

GIVEN

$m = F_{k_1} + F_{k_2} + \dots + F_{k_s} + F_{k_{s+1}}$

IF $k_{s+1} < k_s - 1$ WE'RE DONE; OTHERWISE, REPLACE

$F_{k_s} + F_{k_{s+1}}$ WITH $F_{k_s + 1}$

USING THE FIBONACCI RELATION WHICH SHIFTS k_s UP 1, NOW APPLY INDUCTION.

PIGEONHOLE PRINCIPLE:

GIVEN TWO FINITE SETS

P = PIGEONS

H = HOLES

$f: P \to H$.

IF $|P| > |H|$ THEN FOR SOME $h \in H$, $|f^{-1}(h)| \geq 2$.

EXAMPLE: THERE ARE INTEGERS a, b, c, NOT ALL ZERO $|a|, |b|, |c| < 10^6$, SUCH THAT

$|a + b\sqrt{2} + c\sqrt{3}| < \frac{1}{10^{11}}$.

PROOF:

LET $S = \{a + b\sqrt{2} + c\sqrt{3} : 0 \leq a, b, c < 10^6\}$

ALL OF THESE FALL INTO $[0, (1+\sqrt{2}+\sqrt{3}) \cdot 10^6]$.

SPLIT THIS INTERVAL INTO $10^{18} - 1$ EQUAL SIZE INTERVALS.

SINCE THE NUMBER OF CHOICES OF (a,b,c) IS 10^{18}, TWO LAND IN THE SAME INTERVAL, WHICH HAS LENGTH

$$\frac{10^6(1+\sqrt{2}+\sqrt{3})}{10^{18}-1} < 10^{-11}.$$

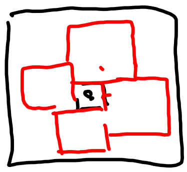
THE DIFFERENCE BETWEEN THESE TWO SUFFICES.

EXTREMAL CONFIGURATIONS:

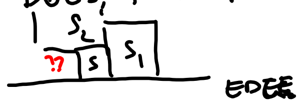
ITS OFTEN USEFUL TO CONSIDER AN EXTREMAL OBJECT.

THEOREM: IT IS NOT POSSIBLE TO DISSECT A CUBE INTO FINITELY MANY CUBES OF DIFFERING EDGE LENGTH.

PROOF: SUPPOSE THE CONTRARY SUCH A DISSECTION EXISTS. CONSIDER BOTTOM FACE, WHICH IS A SQUARE DISSECTED INTO SQUARES OF DIFFERING EDGE LENGTH



LET S HAVE THE SHORTEST EDGE. IT MUST BE THE CASE THAT S DOES NOT TOUCH AN EDGE



LET C BE THE CUBE WITH S AS ITS BOTTOM. SINCE THE CUBES ADJACENT TO S HAVE LARGER EDGE LENGTH THE TOP SURFACE OF C IS A SQUARE WHICH IS DISSECTED. REPEAT THE ARGUMENT WITH THE TOP SURFACE OF C. THIS PROCESS CAN BE CONTINUED FOREVER CONTRADICTION. □

PROBLEMS TO PRESENT:

~~1~~ ~~2~~ 11 12 ~~3~~ 14

① $3^2+4^2=5^2$

$5^2+12^2=13^2$

$c^2+(d-1)^2=d^2$

$c^2=d^2-(d-1)^2=\cancel{d}-\cancel{d}+1\cdot(2d-1)$

$c^2=2d-1$

$\frac{c^2+1}{2}=d$

$\{3^2+4^2=5^2,\ 5^2+12^2=13^2,\ 13^2+\left(\frac{13^2+1}{2}-1\right)^2=\left(\frac{13^2+1}{2}\right)^2 \ \dots$

$\}$

$\forall 2\le i\le k$ & $k=\sum_{1}^{i}\_^2$

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② $n=1$ clearly true.

$\frac{1}{2}=1-\frac{1}{2}=\frac{1}{2}$

Assume true for n.

$p(n)$ $\frac{1}{n+1}+\frac{1}{n+2}+\dots\frac{1}{2n}=1-\frac{1}{2}+\frac{1}{3}\dots-\frac{1}{2n}\to\tilde{p}(n)$

WTS: It's true for $n+1$

$p(n+1)=\tilde{p}(n+1)\iff p(n+1)-p(n)=\tilde{p}(n+1)-\tilde{p}(n)$

$\frac{1}{n+1}+\frac{1}{n+2}\cdots\frac{1}{2(n+1)}-p(n)\overset{?}{=}\frac{1}{2n+1}+\frac{1}{2n+2}-\frac{1}{n+1}$

$RHS=\frac{1}{2n+1}-\frac{1}{2n+2}$

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③ $|\sin nx|\le n|\sin x|$

$n=1$: $|\sin x|\le|\sin x|$

Assume true for n $|\sin nx|\le n|\sin x|$

For $n+1$: WTS: $|\sin(n+1)x|\le(n+1)|\sin x|$

$|\sin((n+1)x)|=|\sin(nx+x)|=|\sin nx\cdot\cos x+\cos nx\cdot\sin x|$

$\le|\sin nx\cdot\cos x|+|\cos nx\cdot\sin x|$

$\le n|\sin x\cos x|+|\sin x|$

$\le n|\sin x|+|\sin x|$

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⑭ $n\ge3$ points plane

sum of angles reg shape

$\theta\le\frac{\pi}{n}$ $\theta\le\frac{\pi(n-2)}{n}$ ← # sides

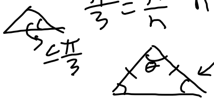
Δ equilateral

Sum of angles in $\Delta=180°$ $=\pi$ rad

$\frac{\pi}{3}=\frac{\pi}{n}$ $n=3$

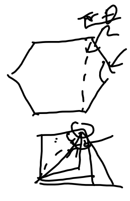
$\frac{\pi-\theta}{2}$ other 2 angles

$\le\frac{\pi}{3}$

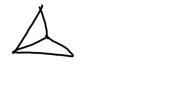


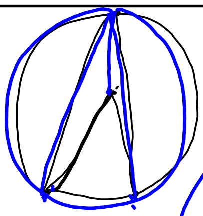
$\frac{1}{2}\left(\pi\left(1-\frac{n-2}{n}\right)\right)$

$\frac{1}{2}\left(\pi\left(\frac{\cancel{n}-\cancel{n}+2}{n}\right)\right)\to\frac{\pi}{n}$

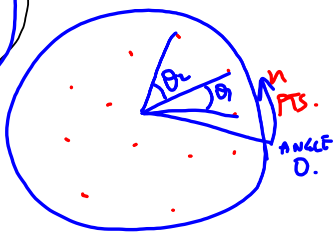


$\frac{\pi(n-2)}{n}$

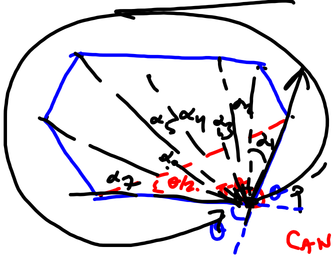




$\frac{\pi}{n}$ $\frac{\pi}{2n} < \frac{\pi}{n+1}$



n POINTS.



$\alpha_1 + \dots + \alpha_{n-2} \le \frac{(n-2)\pi}{n}.$

SOME $\alpha_i \le \pi/n.$

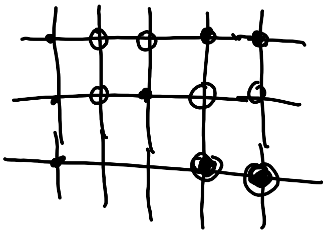
ANGLE IS $\le$

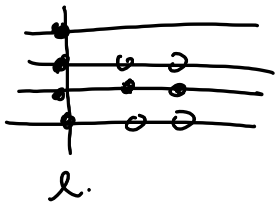
$\pi - \frac{2\pi}{n}.$

$= \frac{(n-2)\pi}{n}.$

P IN CONVEX HULL.

CAN ASSUME $\theta \ge \frac{2\pi}{n}.$





$F_0 = F_1 = 1$, $F_2 = 2$, $F_3 = 3$

$12 = 8 + 3 + 1$ $F_4 = 5$

$F_5 + F_3 + F_1.$ $F_5 = 8$

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n PICK F_m SO THAT

F_m IS LARGEST FIB. # $\le n$.

$n - F_m$. F_ℓ LARGEST FIB

$F_\ell < F_{m-1}$ OR ELSE # $\le n - F_m$.

$F_{m-1} + F_m = F_{m+1} \le n.$

APPLY INDUCTION TO $n - F_m$.