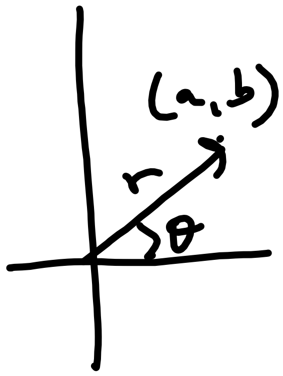
PUTNAM SEMINAR

GEOMETRY AND COMPLEX NUMBERS

A FREQUENTLY USED TRICK ON THE PUTNAM USES COMPLEX NUMBERS TO REPRESENT VECTORS IN 2d.



$a+ib \in \mathbb{C}.$

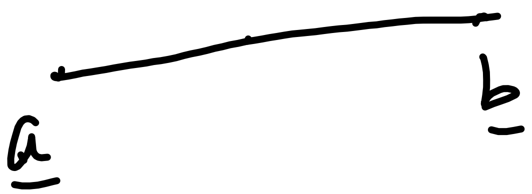
$r = \sqrt{a^2+b^2}$

$\theta = \tan^{-1}\left(\frac{b}{a}\right).$

MULTIPLYING BY $re^{i\theta}$ DILATES BY r AND ROTATES BY θ.

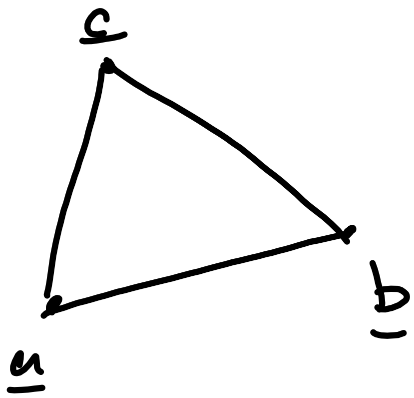
ADDING $a+ib$ PERFORMS A TRANSLATION.

VECTOR GEOMETRY:



SEGMENT: $\underline{a} + t(\underline{b} - \underline{a})$

MIDPOINT: $\dfrac{\underline{a} + \underline{b}}{2}$



INTERIOR OF TRIANGLE IS

$$\{t_1 a + t_2 b + (1-t_1-t_2)c : 0 \le t_1, t_2, 1-t_1 t_2\}.$$

CENTROID: INTERSECTION OF MEDIANS.

THE MEDIAN FROM $\underline{a}$ TO $\overline{bc}$ IS $t(\underline{a}) + (1-t)\frac{\underline{b}+\underline{c}}{2}$

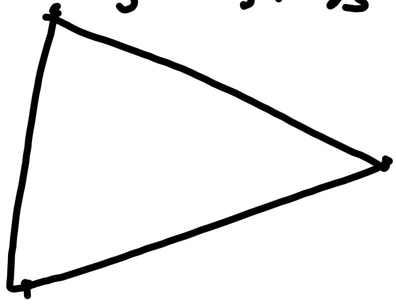
THE MEDIAN FROM $\underline{b}$ TO $\overline{ac}$ IS $t(\underline{b}) + (1-t)\frac{\underline{a}+\underline{c}}{2}$

THE MEDIAN FROM $\underline{c}$ TO $\overline{ab}$ IS $t\underline{c} + (1-t)\frac{\underline{a}+\underline{b}}{2}$.

THE POINT $\frac{\underline{a}+\underline{b}+\underline{c}}{3}$ LIES ON ALL THREE MEDIANS, SO IS THE CENTROID.

TO CALCULATE ALTITUDES

$z_3 = x_3 + iy_3$



$z_2 = x_2 + iy_2$

$z_1 = x_1 + iy_1$

$$\overline{z_2 z_1} = (x_2 - x_1) + (y_2 - y_1)i.$$

ITS PERPENDICULAR IS FOUND BY MULT. BY i:

THE PERPENDICULAR IS

$$-(y_2 - y_1) + i(x_2 - x_1)$$

THE ALTITUDE IS

$$z_3 + t\left(-(y_2 - y_1) + i(x_2 - x_1)\right).$$

$$= z_3 + ti(z_2 - z_1).$$

THE OTHER ALTITUDES ARE

$$z_2 + ti(z_1 - z_3)$$

$$z_1 + ti(z_3 - z_2).$$

A REGULAR n-GON

IS OBTAINED BY TAKING

$1, \zeta, \zeta^2, \ldots, \zeta^{n-1}$; $\zeta = e^{\frac{2\pi i}{n}}$.

PROBLEMS TO PRESENT.

1 2 6 9 13 3

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$$\vec{a}, \vec{b}, \vec{c}$$

$$\vec{n} = (\vec{b} - \vec{a}) \times (\vec{c} - \vec{a})$$

$$\vec{n} = \vec{b} \times \vec{c} - \vec{b} \times \vec{a} - \vec{a} \times \vec{c} + \cancel{\vec{a} \times \vec{a}}$$

$$= \vec{b} \times \vec{c} + \vec{a} \times \vec{b} + \vec{c} \times \vec{a}$$

$\vec{a} + \vec{b} \perp b$

$\vec{a} \times \vec{b} + \cancel{\vec{b} \times \vec{b}} + \vec{c} \times \vec{b}$

$\cancel{-\vec{c} \times \vec{b}}$ $\cancel{+\vec{c} \times \vec{b}}$

$(\vec{a} + \vec{b} + \vec{c}) \times \vec{b} = 0$

$(\vec{a} + \vec{b} + \vec{c}) \times \vec{c} = 0$

$(\vec{a} + \vec{b} + \vec{c}) \times \vec{a} = 0$

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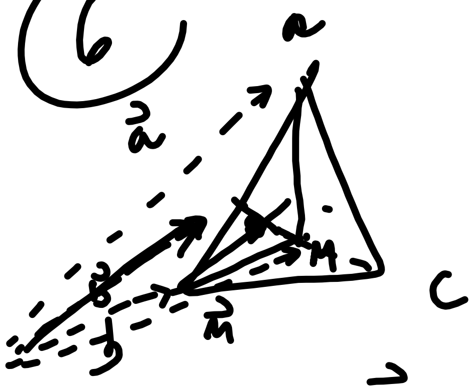


$$\frac{2\left(\frac{\vec{a}+\vec{b}}{2}+\vec{m}\right)}{3}$$

$$\vec{a}-\vec{b}$$
$$\boxed{\vec{b}-\vec{c}}$$
$$\vec{c}-\vec{a}$$

$\frac{1}{3}(\vec{a}+\vec{b}+\vec{M}) -$

$\frac{1}{3}(\vec{a}+\vec{c}+\vec{M})$

$\boxed{= \frac{1}{3}\vec{b} - \frac{1}{3}\vec{c}}$

$\frac{1}{3}(\vec{a}+\vec{b}+\vec{M})$

$\frac{1}{3}(\vec{a}+c+M)$

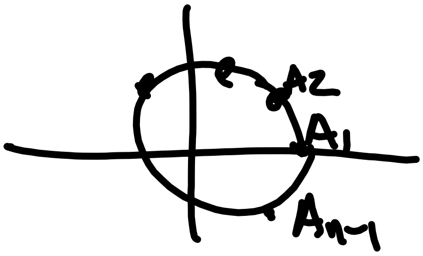
$\frac{1}{3}(\vec{b}+\vec{c}+\vec{M})$

a) $\underline{\zeta}^n + 1 = 0$ n-th

$\zeta, \zeta^2, \dots, \zeta^3$

$$\alpha^n + 1 = 0$$

$$\alpha^0, \alpha^1, \cdots, \alpha^{n-1}$$



$$\prod_{k=1}^{n} PA_k = \prod_{k=1}^{n} |P - A_k|$$

$$= \left|\prod_{k=1}^{n} (P - A_k)\right|$$

$$= \left|\prod_{k=1}^{n} (P - \alpha^k)\right| \qquad x^n - 1 = \prod (x - \alpha^k)$$

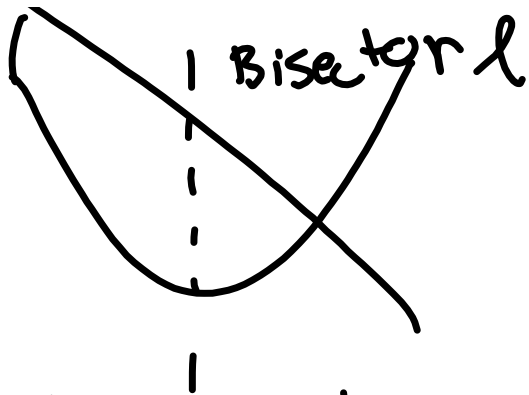
$$= |P^n - 1|$$

$$= \boxed{|p^n - 1|} \leq |p^n| + 1$$

$$\hookrightarrow |-1-1| = 2.$$

Pick p to be n-th root of -1 $\quad p \ (x^n+1)$

$y = ax^2$



Lemma.

Any line not parallel to ℓ intersects the parabola in a bded. set

$$y_1 = ax^2$$

$$y_2 = mx + b$$

$$\lim_{x \to \pm\infty} \frac{y_1}{y_2} = \pm \infty$$

If $P_1, \dots, P_n$ cover $\mathbb{R}^2$, pick ℓ not parallel to any of their bisectors.

$$\underline{\text{bded}} = \ell \cap [P_1 \cup \dots \cup P_n] = \ell$$

$$\ell \cap \mathbb{R}^2 = \ell \quad !$$

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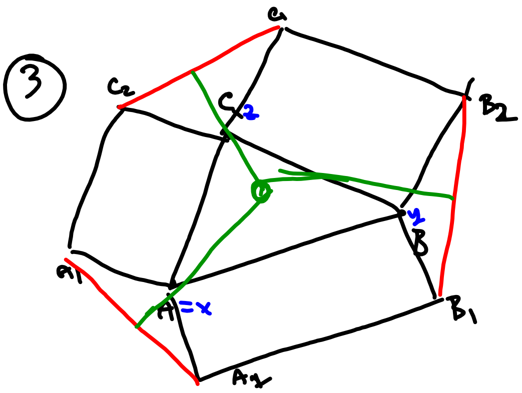

THE DIRECTION PERP TO $\overline{AB}$ IS $AB^\perp$. $A_2 = A + t_1 AB^\perp$, $B_1 = B + t_1 AB^\perp$

$B_2 = B + s\, BC^\perp$, $C_1 = C + s\, BC^\perp$

$C_2 = C + u\, AC^\perp$, $A_1 = A + u\, AC^\perp$.

$$\frac{A_1 + A_2}{2} = A + \tfrac{1}{2}(t\, AB^\perp + u\, AC^\perp)$$

$$\frac{B_1 + B_2}{2} = B + \tfrac{1}{2}(t\, AB^\perp + s\, BC^\perp)$$

$$\frac{C_1 + C_2}{2} = C + \tfrac{1}{2}(s\, BC^\perp + u\, AC^\perp).$$

THE $\perp$ BISECTOR TO A_1A_2 IS

$$\frac{A_1 + A_2}{2} + \alpha (A_2 - A_1)^\perp$$

$$A_2 - A_1 = t\, AB^\perp - u\, AC^\perp$$

$$= \frac{A_1 + A_2}{2} + \alpha (t\, AB - u\, AC).$$

$$= A + \tfrac{1}{2}(t\, AB^\perp + u\, AC^\perp) + \alpha (t\, AB - u\, AC).$$

THE $\perp$ BISECTOR TO B_1B_2 IS

$$= B + \tfrac{1}{2}(t\, AB^\perp + s\, BC^\perp) + \beta (t\, AB + s\, BC)$$

THE $\perp$ BISECTOR TO C_1C_2 IS

$$= C + \tfrac{1}{2}(s\, BC^\perp + u\, AC^\perp). + \gamma (s\, BC + u\, AC).$$

WE WISH TO SHOW THAT WE CAN CHOOSE α, β, γ IN SUCH A WAY THAT THERE IS A COMMON POINT.

SOLVE TWO OF THE EQUATIONS