PUTNAM SEMINAR,
REAL ANALYSIS.

- INTERMEDIATE VALUE THEOREM
- MEAN VALUE THM
- CONVEXITY
- INTEGRATION AND DIFF. OF POWER SERIES TERM-BY-TERM WITHIN THE RADIUS OF CONVERGENCE

INTERMEDIATE VALUE THEOREM:

- IF f IS CONTINUOUS ON $[a,b]$, THEN FOR EACH VALUE c BETWEEN $f(a), f(b)$ THERE IS AN x IN THE INTERVAL WITH $f(x) = c$.

MEAN VALUE THEOREM:

IF f IS DIFF. ON (a,b), CTS ON $[a,b]$ THEN THERE IS $c \in (a,b)$ SUCH THAT

$$f'(c) = \frac{f(b)-f(a)}{b-a}.$$

SLOPE OF TANGENT LINE AT THE POINT.

SLOPE OF SECANT

CONVEXITY:

WE SAY f IS CONVEX ON $[a,b]$ IF, FOR ALL $x, y \in [a,b]$,

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2}.$$

JENSEN'S INEQUALITY APPLIES TO CONVEX FNS:

JONSEN'S INEQUALITY:

IF $t_1, \dots, t_n \geq 0$,

$t_1 + \dots + t_n = 1$ THEN,

IF f IS CONVEX

$$f(t_1 x_1 + \dots + t_n x_n) \leq t_1 f(x_1) + \dots + t_n f(x_n).$$

POWER SERIES:

$\sum a_n x^n$ GENERATING FN

RADIUS OF CONVERGENCE IS DETERMINED BY $\rho = \limsup |a_n|^{\frac{1}{n}}$.

IF $\rho = \infty$, RADIUS $= 0$
$\rho = 0$ RADIUS $= \infty$

IF $\rho \in (0, \infty)$, $R = \frac{1}{\rho}$.

THE POWER SERIES CONVERGES ABSOLUTELY TO AN ANALYTIC FN WITHIN ITS RADIUS OF CONVERGENCE.

IT CAN BE DIFF/INT TERM-BY-TERM

$$\frac{d}{dx} \sum_n a_n x^n = \sum_{n=1}^{\infty} n a_n x^{n-1}.$$

EULER'S FORMULA:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\frac{d}{dx} e^x = \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!} = e^x.$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2n!}$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

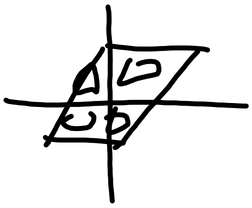
RAD OF CONV. ∞ SINCE $\frac{1}{(n!)^x} \to 0$.

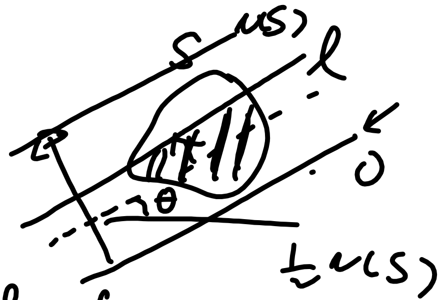
$$e^{ix} = \sum_{n=0}^{\infty} \frac{(ix)^n}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{x^{2n}(-1)^n}{(2n)!} + i\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = \cos x + i \sin x.$$

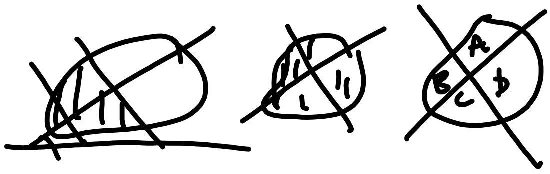
PROBLEMS TO PRESENT:

~~4~~ ~~3~~ ~~2~~ 15

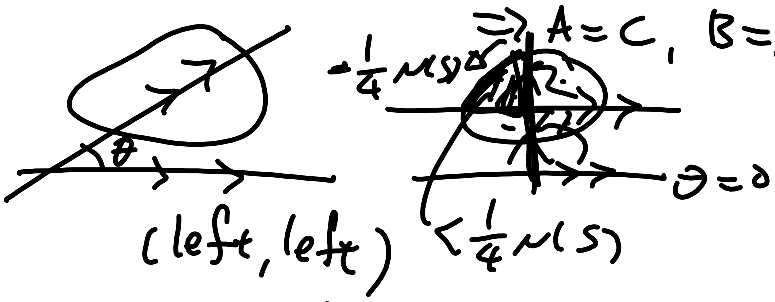




$\ell \leftarrow$ fn of θ

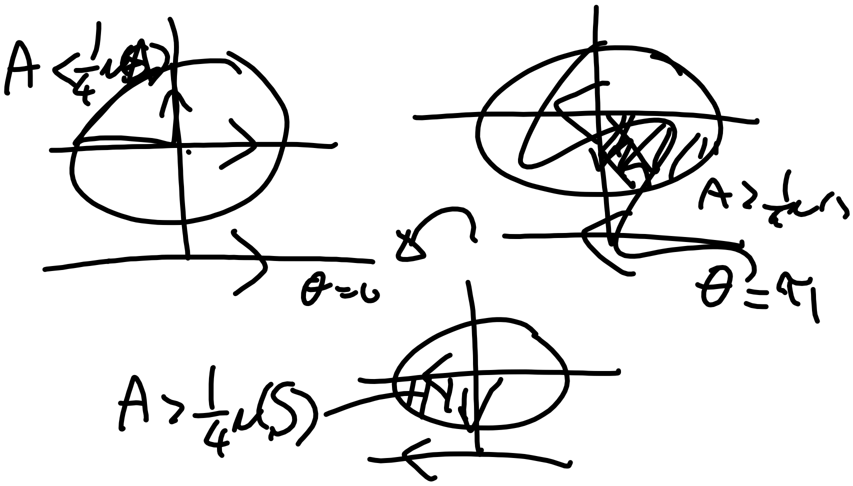


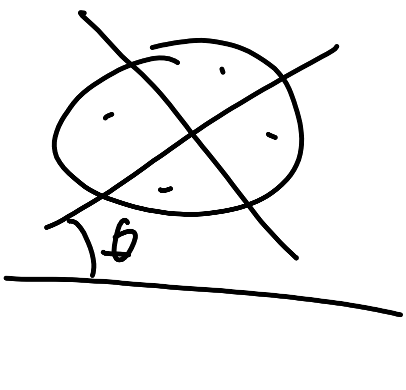
$A+B=C+D$
$A+D=B+C$
$\Rightarrow A=C,\ B=D$



(left, left)

$S(\theta)$ = func that gives the area of the left, left quadr





3 fixed point $f(c) = c$

$f : [a, b] \to [a, b]$ CTS

Let $f(x_1) = a$ min $f(x_2) = b$ max

$g(x) = f(x) - x$

$g(x_1) = a - x_1 \le 0$

$g(x_2) = b - x_2 \ge 0$

$g(x_0) = 0$

$\blacksquare$

$$f(-x) = -f(x)$$

$$\underline{-f(+x)} = f(-x) \le \underline{|\sin(+x)|}$$

$$|f(x)| \le |\sin x|$$

MVT. For $x>0$,

$$\frac{|\sin x|}{|x|} \geq \left|\frac{f(x)}{x}\right| = \left|\frac{f(x)-f(0)}{x-0}\right| = |f'(c)|$$

$\exists\, c \in (0,x)$

$$|f'(c)| \le \left| \frac{\sin x}{x} \right| = RHS$$

$$f'(c) = \sum k a_k \cos(kc)$$

$$\lim_{x \to 0} \left| \sum k a_k \right| \le 1$$

(15) TREAT $C[0,1]$ AS AN INNER PRODUCT SPACE

$$\langle f, g\rangle = \int_0^1 f(x)g(x)\,dx.$$

WE'RE GIVEN $\langle f, 1\rangle = 1$
$\langle f, x\rangle = 1.$

QUESTION: WHAT IS THE MIN. OF $\langle f,f\rangle$?

THE MIN IS ACHIEVED BY A VECTOR IN THE SPAN OF $\{1, x\}$, HENCE A LINEAR FUNCTION.

$$f = a + bx.$$

$$\int_0^1 a + bx\,dx = a + \frac{b}{2} = 1.$$

$$\int_0^1 ax + bx^2\,dx = \frac{a}{2} + \frac{b}{3} = 1.$$

$$\langle f,f\rangle = \int_0^1 (a+bx)^2\,dx = \int_0^1 a^2 + 2bax + b^2x^2\,dx$$
$$= a^2 + ab + \frac{b^2}{3}.$$

MINIMIZE: $a^2 + ab + \frac{b^2}{3}$

SUBJECT TO: $a + \frac{b}{2} = 1$
$\frac{a}{2} + \frac{b}{3} = 1$

$$\frac{b}{2} - \frac{2b}{3} = -1$$
$$-\frac{b}{6} = -1 \qquad b = 6.$$
$$\frac{a}{2} + 2 = 1 \qquad a = -2.$$

ANSWER: $a^2 + ab + \frac{b^2}{3}$ WITH $a=-2$, $b=6$

$$= 4 - 12 + 12 = \boxed{4}.$$

(21) WE WISH TO SHOW:

$$\frac{\pi - x}{2} = \frac{\sin x}{1} + \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \cdots$$

TREAT THIS AS A FUNCTION IN

$L^2[0, 2\pi]$.

$$\left\langle \frac{\pi - x}{2}, 1 \right\rangle = 0.$$

$\cos n(2\pi - x) = \cos nx$. THIS PROVES $\left\langle \frac{\pi - x}{2}, \cos nx \right\rangle = 0$.

TO DETERMINE THE FOURIER SERIES CALCULATE

$$\left\langle \frac{\pi - x}{2}, \sin mx \right\rangle \qquad m \neq 0.$$

$$\int_0^{2\pi} \frac{\pi - x}{2} \cdot \sin mx \, dx = -\frac{1}{2} \int_0^{2\pi} x \sin mx \, dx$$

$u = x \quad dv = \sin mx\,dx$
$du = dx \quad v = -\frac{1}{m}\cos mx$

$$= -\frac{1}{2} \left\{ -x \frac{\cos mx}{m} - \cancel{\int_0^{2\pi} \left(-\tfrac{1}{m}\right) \cos mx \, dx} \right\}_0^{2\pi}$$

$$= + \frac{2\pi}{2m} = \frac{\pi}{m}.$$

$$\langle \sin mx, \sin mx \rangle = \int_0^{2\pi} (\sin mx)^2 \, dx$$

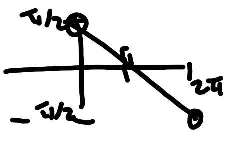
$$= \pi$$

SO

$$\left\{ \frac{1}{\sqrt{\pi}} \sin mx \right\} \text{ ARE O.N.}$$

THIS GIVES THE FOURIER SERIES EXPANSION $\displaystyle \frac{\pi - x}{2} = \sum_{m=1}^{\infty} \frac{\sin mx}{m}$.

AT 0: $\displaystyle \sum_{m=1}^{\infty} \frac{\sin m \cdot 0}{m} = 0$.



$\Rightarrow 0$ AVG. VALUE.

(10) JENSEN'S INEQUALITY:

$$\Leftrightarrow \frac{1}{n}\sum_{i=1}^{n} \log\left(\frac{\sin x_j}{x_j}\right) \le \log\left(\frac{\sin \bar{x}}{\bar{x}}\right)$$

$\bar{x} = \frac{1}{n}\sum x_j$.

SUFFICES $\log \frac{\sin x}{x}$ CONVEX.

TO CHECK THIS CHECK

$$\left(\frac{d}{dx}\right)^2 \log \frac{\sin x}{x} > 0.$$

$$\left(\frac{d}{dx}\right)^2 (\log \sin x - \log x)$$

$$= \frac{d}{dx}\left(\frac{\cos x}{\sin x} - \frac{1}{x}\right)$$

$$= \frac{-\sin^2 x - \cos^2 x}{(\sin x)^2} - \frac{1}{x^2} < 0 \checkmark$$

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$$1-\cos nx = 1-\tfrac{1}{2}(e^{inx}+e^{-inx})$$

$$1-\cos x = 1-\tfrac{1}{2}(e^{ix}+e^{-ix}).$$

$$\frac{1-\cos nx}{1-\cos x} = \frac{\left(e^{\frac{inx}{2}}-e^{-\frac{inx}{2}}\right)^2}{\left(e^{\frac{ix}{2}}-e^{-\frac{ix}{2}}\right)^2}$$

$$\frac{e^{\frac{inx}{2}}-e^{-\frac{inx}{2}}}{e^{\frac{ix}{2}}-e^{-\frac{ix}{2}}} = \qquad z=e^{ix}$$

$$\frac{z^n-1}{z-1}\cdot\frac{z^{-n/2}}{z^{-1/2}}$$

$$=(1+z+z^2+\dots+z^{n-1})$$

$$\int_0^{\pi}\left|(1+z+\dots+z^{n-1})\frac{z^{-n/2}}{z^{-1/2}}\right|^2 d\theta \quad \frac{z^{-n/2}}{z^{-1/2}}.$$

$$=\int_0^{\pi}|1+z+\dots+z^{n-1}|^2\,d\theta$$

$$=\int_0^{\pi}(1+z+\dots+z^{n-1})(1+\bar z+\dots+(\bar z)^{n-1})\,d\theta$$

$$=\int_0^{\pi} z^{-n+1}+2z^{-n+2}+\dots+(n-1)+(n-2)z+\dots+z^{n-1}\,d\theta.$$

$$z=e^{i\theta}=\cos\theta+i\sin\theta$$

$$\int_0^{\pi}\cos\theta\,d\theta=0,\quad \int_0^{\pi}\sin\theta\,d\theta=2.$$

SINCE THE IMAGINARY PART VANISHES, ONLY THE CONSTANT TERM CONTRIBUTES $\Rightarrow \boxed{(n-1)\pi.}$

$$\int_0^{\pi} \frac{1-\cos nx}{1-\cos x}\,dx \Rightarrow -\int_{\pi}^{0} \frac{1-\cos n(\pi-x)}{1-\cos(\pi-x)}\,dx$$

Let $u=\pi-x$

$$\frac{dy}{dx}=-1 \Rightarrow$$

$$\Downarrow$$

$$\int_0^{\pi} \frac{1+(\cos nx)}{1+\cos x}\,dx$$

$$\Rightarrow \int_0^{\pi} \frac{(1-\cos nx)(1+\cos x)+(1+\cos nx)(1-\cos x)}{(1-\cos x)(1+\cos x)}$$

$$\int_0^{\pi} \frac{(1-\cos nx)(1+\cos x)+(1+\cos nx)(1-\cos x)}{(1+\cos x)(1-\cos x)}\,dx$$

$$(1+\cos x)(1-\cos x) \to 1-\cos^2 x$$

$$= \int_0^{\pi} \frac{2-2\cos x\cos nx}{\sin^2 x}\,dx$$

$$\frac{1}{2}\cdot 2\left[\int_0^{\pi}\frac{1}{\sin^2 x}\,dx - \int_0^{\pi}\underline{\cot x\cos x\,dx}\right]$$

$\Downarrow$

$\cancel{\csc^2 x}$ $\int \cancel{\sec}^2 x\,dx = \tan x\Big|_0^{\pi} + \int_0^{\pi}\cot x \cos$

$\int \csc^2 x\,dx = \tan x$

$$\int_0^\pi \frac{2-2\cos x \cos nx}{\sin^2 x}\,dx$$

$$= \int_0^\pi \frac{1}{\sin^2 x}\,dx - \int_0^\pi \underline{\cot x \cos nx\,dx}$$