PUTNAM SEMINAR

SEQUENCES AND SERIES.

USEFUL TRICKS:

RECURRENCES:

A SEQUENCE $a_0, a_1, a_2, \ldots$

IS DEFINED BY

(1) GIVING SOME INITIAL VALUES

(2) DESCRIBING HOW A FURTHER TERM CAN BE GIVEN IN TERMS OF PREVIOUS ONES

$a_n = f(a_0, a_1, \ldots, a_{n-1})$.

- THIS IS ANALOGOUS TO STUDYING DIFFERENTIAL EQUATIONS IN THE CONTINUUM.

LINEAR RECURRENCES:

$$a_n = b_1 a_{n-1} + b_2 a_{n-2} + \cdots + b_k a_{n-k}.$$

STANDARD METHOD OF SOLUTION:

INTRODUCE THE GENERATING FUNCTION

$$f(x) = \sum_{n=0}^{\infty} a_n x^n.$$

(IN THE CASE OF LINEAR ODE, INSTEAD TAKE LAPLACE TRANSFORM).

SUBSTITUTE RECURRENCE

$$\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{k-1} a_n x^n + \sum_{n=k}^{\infty} (b_1 a_{n-1} + b_2 a_{n-2} + \dots + b_k a_{n-k}) x^n .$$

BY LINEARITY:

$$\sum_{n=0}^{k-1} a_n x^n + b_1 \sum_{n=k-1}^{\infty} a_n x^{n+1} + b_2 \sum_{n=k-2}^{\infty} a_n x^{n+2} + \dots + b_k \sum_{n=0}^{\infty} a_n x^{n+k} .$$

$$= (b_1 x + b_2 x^2 + \dots + b_k x^k) f(x) + Q(x)$$

$Q(x)$ IS A POLYNOMIAL.

$$f(x)(1 - b_1 x - b_2 x^2 - \dots - b_k x^k) = Q(x)$$

$$f(x) = \frac{Q(x)}{(1 - b_1 x - \dots - b_k x^k)}$$

DEVELOP RHS IN PARTIAL FRACTIONS.

$$f(x) = \frac{c_0 + c(x-x_0) + \dots + c_{n_1-1}(x-x_0)^{n_1-1}}{(x-x_0)^{n_1}} + \frac{d_0 + d_1(x-x_1) + \dots + d_{n_2-1}(x-x_1)^{n_2-1}}{(x-x_1)^{n_2}} \dots + \dots$$

THIS IS THE GENERAL SOLUTION OF A LINEAR RECURRENCE.

USE GEOMETRIC SERIES TO GIVE A CLOSED FORM FOR THE TERMS

$$\frac{1}{1-x} = 1 + x + x^2 + \dots$$

$$\frac{d}{dx} \frac{1}{1-x} = \frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \dots$$

$$\left(\frac{d}{dx}\right)^2 \frac{1}{1-x} = \frac{2}{(1-x)^3} = 2 + 6x + 12x^2 + \dots$$

$\vdots$

EXAMPLE: FIBONACCI NUMBERS

$F_0 = F_1 = 1$

$F_n = F_{n-1} + F_{n-2}.$

$$\sum_{n=0}^{a} F_n x^n = 1 + x + \sum_{n=2}^{\infty} (F_{n-2} + F_{n-1}) x^n$$

$$= (1+x) + \sum_{n=1}^{\infty} F_n x^{n+1} + \sum_{n=0} F_n x^{n+2}$$

$$= 1 + x f(x) + x^2 f(x)$$

$$(1 - x - x^2) f(x) = 1$$

$$f(x) = \frac{1}{1 - x - x^2}.$$

$$\frac{1}{\left(1 - \frac{1+\sqrt{5}}{2} x\right)\left(1 - \frac{1-\sqrt{5}}{2} x\right)}.$$

$$= \frac{A}{1 - \frac{1+\sqrt{5}}{2} x} + \frac{B}{1 - \frac{1-\sqrt{5}}{2} x} = 1.$$

$$A\left(1 - \frac{1-\sqrt{5}}{2} x\right) + B\left(1 - \frac{1+\sqrt{5}}{2} x\right) = 1 - x - x^2$$

PLUG IN $x = \frac{\sqrt{5}-1}{2}$

$$A \cdot \left(1 + \left(\frac{\sqrt{5}-1}{2}\right)^2\right) = 1 - \left(\frac{\sqrt{5}-1}{2}\right) - \left(\frac{\sqrt{5}-1}{2}\right)^2.$$

SOLVE FOR A, THEN B SIMILARLY.

THIS EXPRESSES

$$F_n = A\left(\frac{1+\sqrt{5}}{2}\right)^n + B\left(\frac{1-\sqrt{5}}{2}\right)^n$$

DON'T BE SHY ABOUT USING
TAYLOR EXPANSION TO
CHECK WHICH TERMS MATTER
SEE E.G. #7 ON #9
HANDOUT, #10.

TELESCOPING SUMS:

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

$$\frac{1}{n(n+1)} \to \frac{1}{n} - \frac{1}{n+1}$$

$$\left(1 - \cancel{\tfrac{1}{2}}\right) + \left(\cancel{\tfrac{1}{2}} - \cancel{\tfrac{1}{3}}\right) + \left(\cancel{\tfrac{1}{3}} - \cancel{\tfrac{1}{4}}\right) + \cancel{\cdots} = 1.$$

THIS IS CLOSELY RELATED TO ABEL SUMMATION OR SUMMATION BY PARTS.

LET $A_n = \sum_{k=0}^{n} a_k$

$B_n = \sum_{k=0}^{n} b_k$

$\sum A_n b_n = \sum A_n (B_n - B_{n-1})$

$\sum A_n B_n - A_n B_{n-1}$

$= \sum -B_n A_{n+1} + B_n A_n$ + INITIAL TERM

$= -\sum B_n a_{n+1}$ + INITIAL TERM.

PROBLEMS TO PRESENT

~~#2~~ ~~#4~~ ~~#9~~ ~~#1~~

#3 #14

#2

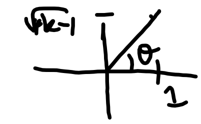
$$1-\binom{n}{2}a+\binom{n}{4}a^2-\binom{n}{6}a^3+\cdots$$

$$\mathrm{Re}\left((1+i\sqrt{a})^n\right).$$

$$z^n \qquad z=Re^{i\theta}$$

$$\mathrm{Re}(R^n e^{in\theta}) = R^n\cos(n\theta).$$

$$R=\sqrt{1+a} = 2\sqrt{k}$$



$$\theta = \tan^{-1}\sqrt{4k-1}.$$

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$$

$$\cos n\theta = \frac{e^{in\theta}+e^{-in\theta}}{2}$$

$$e^{in\theta}+e^{-in\theta} = P_n(\underbrace{e^{i\theta}+e^{-i\theta}}_{=\,2\cos\theta}).$$

$$\left(z+\frac{1}{z}\right)^n = z^n+\binom{n}{1}z^{n-2}+\binom{n}{2}z^{n-4}+\cdots+\binom{n}{n}z^{-n}.$$

$$P_n(x) = x^n - nx^{n-2} + \underbrace{\binom{n}{2}}_{\binom{n}{2}-n\cdot\binom{n-1}{1}}x^{n-4}+\cdots$$

INTEGER COEFFICIENTS.

$$R\cos\theta = 1,\quad R=2\sqrt{k}$$

DESIRED QUANTITY:

$$R^n\cdot\frac{P_n(2\cos\theta)}{2}.$$

$$\downarrow$$

$$2^n k^{n/2}\cdot\frac{P_n(2\cos\theta)}{2} = 2^{n-1}k^{\frac{n}{2}}P_n(2\cos\theta)$$

$$2\cos\theta = \frac{1}{\sqrt{k}}$$

$$P_n\left(\frac{1}{\sqrt{k}}\right) = \left(\frac{1}{\sqrt{k}}\right)^n + c_1\left(\frac{1}{\sqrt{k}}\right)^{n-2} + c_2\left(\frac{1}{\sqrt{k}}\right)^{n-4}+\cdots$$

$$k^{\frac{n}{2}}P_n\left(\frac{1}{\sqrt{k}}\right) = 1+c_1k+c_2k^2+\cdots \in \mathbb{Z}. \quad \square$$

#4 $\lim_{n\to\infty} \left|\sin\left(\pi\sqrt{n^2+n+1}\right)\right|$

$$\left(\sqrt{n^2+n+1}-n\right)\left(\sqrt{n^2+n+1}+n\right) = \cancel{n^2}+n+1-\cancel{n^2}$$

$$\sqrt{n^2+n+1}-n = \frac{n+1}{\sqrt{n^2+n+1}+n}$$

$$\lim_{n\to\infty} \frac{n+1}{\sqrt{n^2+n+1}+n} = \frac{1}{2}$$

$$\sin\left(\pi\sqrt{n^2+n+1} - n\pi\right) = \sin$$

$$\sin\left(\pi\left(\sqrt{n^2+n+1} - n\right)\right)$$

$$\sin \pi/2 = 1$$

$$\sqrt{1+\sqrt{1+\sqrt{1+\cdots}}}$$

$$x = \sqrt{1+x}$$

x_n

$$x_n : \sqrt{1+x_{n-1}}$$

$x^2 = 1+x$

$$x_n = \sqrt{1+x_{n-1}}$$

$$\log x_n = \frac{1}{2}\log(1+x_{n-1})$$

$$x^2 - x - 1 = 0$$

$$\frac{1 \pm \sqrt{5}}{2}$$

$$\frac{1+\sqrt{5}}{2}$$

$$x_n = \sqrt{1+x_{n-1}}$$

WISH TO SHOW

$$\lim_{n\to\infty} x_n = \frac{1+\sqrt{5}}{2}.$$

$$x_n - \frac{1+\sqrt{5}}{2} = \sqrt{1+x_{n-1}} - \left(\frac{1+\sqrt{5}}{2}\right)$$

$$= \frac{1+x_{n-1} - \left(\frac{1+\sqrt{5}}{2}\right)^2}{\sqrt{1+x_{n-1}} + \frac{1+\sqrt{5}}{2}}.$$

$$\left(\frac{1+\sqrt{5}}{2}\right)^2 = \frac{6+2\sqrt{5}}{4}$$

$$= \frac{3+\sqrt{5}}{2} = 1 + \frac{1+\sqrt{5}}{2}$$

$$= \frac{x_{n-1} - \left(\frac{1+\sqrt{5}}{2}\right)}{x_{n-1}+1+\frac{1+\sqrt{5}}{2}} \quad > 1. \quad \to 0 \text{ AS } n\to\infty.$$

#1

$1+2+3+4\cdots$

$\frac{n(n+1)}{2}$

1 2 2 3 3 3

$t_n:\ j \quad \frac{j(j-1)}{2} < n \leq \frac{j(j+1)}{2}$

$n \in (0, 1]$ $a_n = 1$

$n \in (1, 3]$ $a_n = 2$

$\vdots$

$n \in \left(\frac{i(i-1)}{2}, \frac{i(i+1)}{2}\right]$ $a_n = i$

$n > \frac{i^2 - i}{2}$ (1) $\quad n \leq \frac{i(i+1)}{2}$

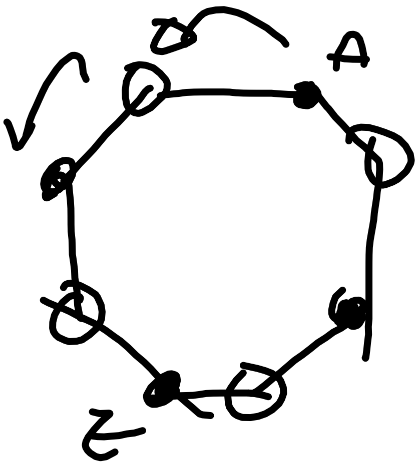
$i^2 - i - 2n < 0 \quad i^2 + i - 2n \geq 0$

$i \in \left(\frac{1-\sqrt{1+8n}}{2}, \frac{1+\sqrt{1+8n}}{2}\right)$ $i \in (-\infty, \frac{-1-\sqrt{1+8n}}{2}] \cup [\frac{-1+\sqrt{1+8n}}{2}, +\infty)$

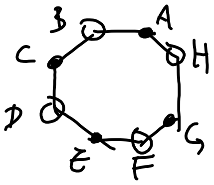
$i \in \left[\frac{-1+\sqrt{1+8n}}{2}, \frac{1+\sqrt{1+8n}}{2}\right)$

$a_n = \left\lceil \frac{\sqrt{1+8n} - 1}{2} \right\rceil \in \mathbb{N}$

$a_n = \left\lfloor \frac{1+\sqrt{1+8n}}{2} \right\rfloor$ $\quad \frac{5-1}{2}$ $\quad 2$



$a_{2n-1} = 0$



① $A \to C \rightsquigarrow E$

② $A \to G \rightsquigarrow E$

③ $A \to B \to A$

$A \to H \to A$

$\#(2n\text{-length from } A \to E)$

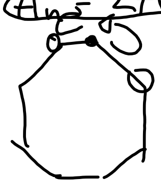
$= \#(2n-2 \text{ length } C \to E)$

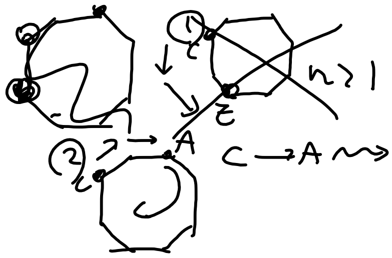
$+ \#(2n-2 \sim\sim A \to E)$

$+ \#(2n-2 \sim\sim G \to E)$ B_{n-1}

$a_{2n} = A_n$

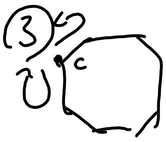
$A_n = 2A_{n-1} + 2B_{n-1}$





$C \to A \rightsquigarrow E$

③



$C \to B \to C$

$C \to D \to C$

$B_n = A_{n-1} + 2B_{n-1}$

$$\begin{cases} A_n = 2A_{n-1} + 2B_{n-1} \\ B_n = A_{n-1} + 2B_{n-1} \end{cases}$$

$$\begin{pmatrix} A_n \\ B_n \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} A_{n-1} \\ B_{n-1} \end{pmatrix}$$

$2 \pm \sqrt{2}$

$2+\sqrt{2} \to \begin{pmatrix} 1 \\ \frac{1}{\sqrt{2}} \end{pmatrix}$

$2-\sqrt{2} \to \begin{pmatrix} 1 \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$

$\frac{1}{\sqrt{2}}(x^{n-1} - y^{n-1})$ ///

#H4

$$\zeta^{-1} = \sum_{n=0}^{\infty} \zeta^n (1-\zeta)(1-\zeta^2) \cdots (1-\zeta^n)$$

ζ IS A ROOT OF UNITY,

$\zeta^k = 1$.

NOTE, THE SUM IS FINITE,

ANY TERM WITH $n \geq k$ IS 0.

$$\iff 1 = \sum_n \zeta^{n+1} (1-\zeta)(1-\zeta^2) \cdots (1-\zeta^n).$$

$$\zeta^{n+1} = 1 - (1-\zeta^{n+1}).$$

$$= \sum_n \underbrace{(1-\zeta)\cdots(1-\zeta^n)}_{a_n} \underbrace{- (1-\zeta)\cdots(1-\zeta^{n+1})}_{a_{n+1}}.$$

THIS TELESCOPES, SO EQUAL TO 1.

$$\sum_{n=0}^{\infty} a_n - a_{n+1} = a_0 = 1$$

$a_j = 0$. ALL $j \geq k$.