PUTNAM SEMINAR

ALGEBRA AND LINEAR ALGEBRA

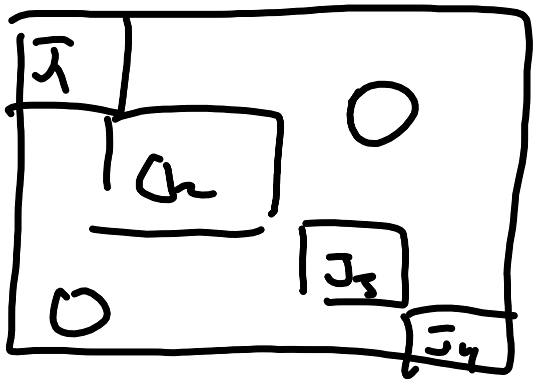
GIVEN A MATRIX A,
WE SAY B IS EQUIVALENT TO A IF

$$B = C^{-1} A C.$$

THIS AMOUNTS TO CHANGE OF BASIS.

JORDAN NORMAL FORM:

EVERY MATRIX A IS EQUIVALENT OVER THE COMPLEX NUMBERS TO A MATRIX OF FORM



J_i IS A JORDAN BLOCK

$$J_i = \begin{pmatrix} \lambda & 1 & & 0 \\ & \lambda & 1 & \\ & & \ddots & \ddots \\ 0 & & & \lambda \end{pmatrix}.$$

THE VALUES λ ARE CALLED THE GENERALIZED EIGENVALUES.

THE <u>DETERMINANT</u> IS SEPARATELY LINEAR IN EACH ROW AND COLUMN, FLIPS SIGN IF YOU EXCHANGE ROWS OR COLUMNS.

$$\det(A \cdot B) = \det(A) \cdot \det(B).$$

THUS IF

$A = C^{-1}BC$ THEN

$\det(A) = \det(B)$.

$$\det\begin{pmatrix} \lambda & & 0 \\ & \lambda & \\ 0 & & \ddots \\ & & & \lambda \end{pmatrix}_{m \times m} = \lambda^m.$$

CHARACTERISTIC POLY:

$p(z) = \det(\underset{n \times n}{A} - zI)$. deg n POLY.

THE # OF ROOTS AT λ IS THE NUMBER OF GENERALIZED EIGENVALUES

$TR(A) = \sum A_{ii}$

SUM OF DIAGONAL ELEMENTS.

$Tr(AB) = Tr(BA)$: TO CHECK

THIS, $C = AB$, $C_{ii} = \sum_j A_{ij} B_{ji}$

SO $\sum (AB)_{ii} = \sum_{i,j} A_{ij} B_{ji} = \sum (BA)_{ii}$

$$\mathrm{Tr}(C^{-1}AC) = \mathrm{Tr}(C \cdot C^{-1} A)$$
$$= \mathrm{Tr}(A)$$

So $\mathrm{Tr}(A) = \mathrm{Tr}(B) = \sum \lambda_i$

SUM OF EIGENVALUES CTED WITH MULT.

$$B = C^{-1}AC$$

$$B^n = (C^{-1}A\cancel{C})(\cancel{C^{-1}}AC)\cdots\cdots(C^{-1}AC)$$

$$= C \cdot A^n \cdot C^{-1}$$

$$\begin{pmatrix} \lambda & 1 & & 0 \\ & \lambda & 1 & \\ & & \ddots & 1 \\ & & & \lambda \end{pmatrix}^n = \left[(\lambda \cdot I) + \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ & & & 0 \end{pmatrix} \right]^n$$

$$= \sum_{k=0}^{n} \binom{n}{k} \lambda^k \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix}^{n-k}.$$

$$\begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & & 0 \\ & 0 & 0 & 1 & \\ & & \ddots & \ddots & 1 \\ & & & 0 & 0 \\ & & & & 0 \end{pmatrix}$$

$\begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix}^{n-k}$ HAS 1'S ON THE $n-k$ DIAGONAL.

$$e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!}$$

$$C^{-1} e^B C.$$

$$e^{\begin{pmatrix} \lambda & 1 & & \\ & \ddots & \ddots & \\ & & & \lambda \end{pmatrix}} = \ldots \sum_{k=0}^{\infty} \frac{(\lambda I + N)^k}{k!}$$

$$= \sum_{k=0}^{\infty} \sum_{m=0}^{k} \binom{k}{m} \cdot N^m \frac{\lambda^{k-m}}{k!}.$$

$$= \sum_{m=0}^{\infty} N^m \sum_{k=m}^{\infty} \frac{1}{m! \cdot (k-m)!} \lambda^{k-m}$$

$$= \sum_{m=0}^{\infty} \frac{N^m}{m!} \cdot e^{\lambda}.$$

VAN DER MONDE:

$$\det\begin{pmatrix} 1 & 1 & & 1 \\ x_1 & x_2 & \cdots & x_n \\ \vdots & \vdots & & \vdots \\ x_1^{n-1} & x_2^{n-1} & & x_n^{n-1} \end{pmatrix}.$$

THIS IS A DEGREE $1+2+\cdots+n-1 = \frac{(n-1)(n)}{2}$

POLYNOMIAL IN $x_1, \ldots, x_n$.

IF $x_i = x_j$, POLY IS ZERO.

THUS IT HAS A FACTOR

$\prod_{i<j}(x_j - x_i)$. THIS HAS DEGREE $\frac{n(n-1)}{2}$

THUS EQUAL UP TO A CONSTANT.

TO DETERMINE THE CONSTANT SUBTRACT x_1 TIMES EACH ROW FROM NEXT:

$$\begin{pmatrix} 1 & * & * & \cdots & * \\ 0 & & & & \\ 0 & & x_j^k - x_1 x_j^{k-1} & & \\ \vdots & & & & \\ 0 & & & & \end{pmatrix} = \prod_{j=2}^{n}(x_j - x_1)\det\begin{pmatrix} 1 & 1 & & 1 \\ x_2 & x_3 & & \vdots \\ \vdots & \vdots & & \\ x_2^{n-2} & x_3^{n-2} & & x_n^{n-2} \end{pmatrix}$$

BY INDUCTION,

$$\prod_{i<j}(x_j - x_i).$$

$$\det \begin{pmatrix} x_1^{j_1} & x_2^{j_1} & \cdots & x_n^{j_1} \\ x_1^{j_2} & x_2^{j_2} & & x_n^{j_2} \\ \vdots & \vdots & & \vdots \\ x_1^{j_n} & x_2^{j_n} & & x_n^{j_n} \end{pmatrix} = P_{j_1,\dots,j_n}(\underline{x})$$

THIS AGAIN HAS A FACTOR OF $\prod_{i<j} (x_j - x_i)$

$$\frac{P_{j_1,\dots,j_n}(\underline{x})}{\prod_{i<j} x_j - x_i}$$

THIS BECOMES A SYMMETRIC FUNCTION

THIS IS A COMMON METHOD OF DEVELOPING THE SYMMETRIC FUNCTIONS.

ENJOY PIZZA AND

PROBLEMS!

PROBLEMS TO PRESENT:

12 FROM LAST WEEK.

3 5 21 6

(12) FR

WEEK!

$$x^{p-1} + 2x^{p-2} + \cdots + (p-1)x + p = P(x)$$

p PRIME IS IRREDUCIBLE OVER $\mathbb{Z}$.

SUPPOSE FOR CONTRADICTION $P(x) = Q(x)R(x)$. INT COEFF.

BOTH Q, R ARE MONIC, LAST COEFF. MULT. TO P SO

$Q(0)R(0) = p \Rightarrow |Q(0)|$ OR $|R(0)| = 1$

INSTEAD CONSIDER

$$px^{p-1}+(p-1)x^{p-1}+\cdots+1$$

(REPLACE $x \to \frac{1}{x}$ MULT BY x^{p-1})

THIS IS

$$\frac{d}{dx}\left(x^{p}+x^{p-1}+\cdots+x+1\right).$$

$$\frac{x^{p+1}-1}{x-1}$$ ALL ROOTS AT

$p+1$ ROOTS OF UNITY.

THUS ROOTS OF

$px^{p-1}+\cdots+1$ ARE ALL <u>INSIDE</u>

UNIT CIRCLE BY GAUSS-LUCAS.

$$x^{p-1}+2x^{p-2}+\cdots+(p-1)x+p$$

ALL ROOTS OUTSIDE UNIT CIRCLE, SO SAME TRUE OF Q, R.

SINCE BOTH MONIC,

$Q(0), R(0)$ ARE SIZE OF PRODUCT OF ROOTS > 1, $\times$ $\square$

3 AND 5

$$\mathrm{tr}(AA^t + BB^t) = \mathrm{tr}(AB + A^tB^t)$$

$$\Leftrightarrow \mathrm{tr}(AA^t + BB^t - AB - A^tB^t)$$

$$= 0$$

$$A(A^t - B) - (A^t - B)B^t$$

$$\operatorname{tr} A(A^t - B) - \operatorname{tr}(A^t - B)B^t$$

$$= \operatorname{tr}\big((A - B^t)(A^t - B)\big) = 0$$

C $\qquad$ C^t

$$\mathrm{tr}(CC^t) = 0$$

$$\Uparrow$$

$$\sum_{i,j} c_{ij}^2 = 0$$

$$\Leftrightarrow c_{ij} = 0$$

$$\Leftrightarrow A - B^t = 0 \quad \square$$

$\prod_{i<j}(x_j - x_i)$ $\quad 1!\,2!\cdots(n-1)!$

$x^m, \binom{x}{m-1}, \binom{x}{m-2}, \ldots, \binom{x}{0}$

P_k, P_k is the space of polys. deg $\leq k$.

$$\binom{x}{m} = a_0\binom{x}{0} + a_1\binom{x}{1} + \cdots + a_{m-2}\binom{x}{m-2} + a_{m-1}\binom{x}{m-1} + a_m x^m.$$

$$\begin{pmatrix} \binom{x_1}{0} & \cdots & \binom{x_n}{0} \\ \vdots & \ddots & \vdots \\ \binom{x_1}{n-1} & \cdots & \binom{x_n}{n-1} \end{pmatrix}$$

$a_{n-1} x_1^n + \cdots$

a_n

$$a_n \left(x_1^n \quad \cdots \quad x_{n-1}^n \right)$$

$$\left(\frac{1}{n-1!} \quad \frac{1}{1! \cdots (n-1)!} \right) (\text{Vand. det})$$

$x^2 = x$

$x^6 - x = 0 \Rightarrow \underline{x^6 - x}$ is the ...

$2^6 - 2 = 3^6 - 3 \Longrightarrow \underline{\gcd(2^6-2, 3^6-\cancel{2})} = \cancel{0}\ 0$

$a = \underset{\sim}{s}(2^6-2) + \underset{\sim}{t}(3^6-2) = \overset{a}{0}$ $\quad 2$

$2 = 0$

$(x+1)^6 = (x+1)$ $\quad$ LHS $= \sum_k \binom{6}{k} x^{6-k}$

$= \cancel{x^6} + \cancel{6x^5} + \underline{15x^4} + \cancel{20x^3} + \underline{15x^2} + \cancel{6x} + x = x+$

$x^4 + x^2 = 0 \quad x^4 + x^2 = \underline{2x^2} = x^4 - x^2$

$\Rightarrow x^4 - x^2 = 0 \Leftrightarrow x^4 = x^2$

$x^6 - x^4 = 0 \Rightarrow x^6 = x^4 = \underline{x^2}$

$\underline{= x}$

$x^2 = x$

$A, B \quad \det A = \det B$

$= \det(A+B) = \det(A-B)$

$= 0$

$A = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} b_1 & 0 & 0 \\ 0 & 0 & b_3 \\ 0 & 0 & 0 \end{pmatrix}$

x, y

$\det(xA + yB) = 0$

$\cancel{ax^3} + bx^2y + cxy^2 + \cancel{dy^3}$

$by + cy^2$

$f(x,y) = \det(xA + yB)$

$A + yB$

(19) CONDITION FOR AN INTEGER n TO START WITH DIGIT 7 IS, FOR SOME m,

$$7\cdot 10^m \le n < 8\cdot 10^m.$$

$$\log_{10} 7 + m \le \log_{10} n < \log_{10} 8 + m.$$

$(\log_{10} n)$ = FRACTIONAL PART $= \log_{10} n - \lfloor \log_{10} n \rfloor$

CONDITION:

$$\boxed{\log_{10} 7 \le (\log_{10} n) < \log_{10} 8.}$$

$$\log_{10} 2^k = k\cdot \log_{10} 2.$$

CONDITION: FOR INFINITELY MANY k, $(\log_{10} 2)\cdot k$ HAS FRACTIONAL PART BETWEEN $\log_{10} 7$ AND $\log_{10} 8$.

THERE ARE A HANDFULL OF WAYS OF PROVING THIS.

$\log_{10} 2$ IS IRRATIONAL.

FACT: IF $0 \le \alpha < \beta < 1$ AND IF γ IS IRRATIONAL,

$$\lim_{n\to\infty} \frac{1}{n} \#\{1 \le m \le n : (m\gamma) \in [\alpha, \beta)\} \to \beta - \alpha.$$

EASIER PROOF THAT THIS HAPPENS ∞-OFTEN: IF $\delta < \beta - \alpha$, THEN SOME MULTIPLE OF δ FALLS IN (α, β), I.E. TAKE THE LEAST MULTIPLE OF $\delta > \alpha$. THIS MEANS THAT WE JUST NEED TO PRODUCE MULTIPLES OF $\log_{10} 2$ WHICH HAVE FRACTIONAL PART $< \log_{10} 8 - \log_{10} 7$.

THIS CAN BE ACCOMPLISHED BY PIGEONHOLE

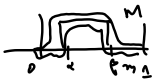
SINCE TWO OF

$0, \log_{10} 2, 2\cdot\log_{10} 2, \dots, n\cdot\log_{10} 2$

HAVE FRACTIONAL PART DIFF. BY $< \frac{1}{n}$, WHICH CAN BE MADE ARB. SMALL.

TO PROVE THE ASYMPTOTIC STATEMENT:

WISH TO COUNT
$$\sum_{1 \le n \le N} \mathbb{1}(n \cdot \log_{10} 2 \in [\alpha, \beta)).$$



IT IS POSSIBLE TO APPROX. $\mathbb{1}_{[\alpha,\beta)}$ FROM ABOVE AND BELOW WITH TRIGONOMETRIC POLYNOMIALS WITH ARBITRARY SMALL L^1 ERROR, SINCE THE TRIGONOMETRIC POLYS ARE DENSE

$$M(x) = \sum_{j=-N}^{N} a_j e^{2\pi i j \cdot x}$$
$$m(x) = \sum_{j=-N}^{N} b_j e^{2\pi i j \cdot x}$$
FINITE.

TREAT THESE TWO POLYS AS FIXED AND LET $N \to \infty$.

$$\frac{1}{N} \sum_{n=1}^{N} m(n \cdot \log_{10} 2) \le \frac{1}{N} \sum_{n=1}^{N} \mathbb{1}_{[\alpha,\beta)}(n \log_{10} 2) \le \frac{1}{N} \sum_{n=1}^{N} M(n \cdot \log_{10} 2)$$

ESTIMATE LEFT, RIGHT SUM.

$$\frac{1}{N} \sum_{n=1}^{N} \sum_{j=-M}^{M} \underbrace{e^{2\pi i j \cdot n \log_{10} 2}}_{1} \cdot a_j .$$

IF $j=0$, PHASE CONSTANT

$$\Rightarrow a_0 .$$

$$j \ne 0, \ \frac{1}{N} \sum_{n=1}^{N} e^{(2\pi i j \cdot \log_{10} 2) \cdot n}$$
← GEOM SERIES.

$$= \frac{1}{N} \frac{e^{2\pi i j \log_{10} 2 \cdot (N+1)} - e^{2\pi i j \log_{10} 2}}{e^{2\pi i j \cdot \log_{10} 2} - 1}$$

$$| \ | \le \frac{2}{N} \cdot \frac{1}{|e^{2\pi i j \log_{10} 2} - 1|}$$ ← CONSTANT

$\to 0$ AS $N \to \infty$.

THUS $\lim$ AS $N \to \infty$ IS a_0.
SIMILARLY b_0 FOR RHS.

$$a_0 \le \lim_{N \to \infty} \frac{1}{N} \# \{ j \cdot \log_{10} 2 \in [\alpha, \beta) : 1 \le j \le N \} \le b_0 .$$

$a_0 = \int m(x) dx$
$b_0 = \int M(x) dx$, THESE CAN BE MADE ARB. CLOSE TO $\beta - \alpha \Rightarrow$ LIMIT.

THIS PROVES THE WEAK LIMIT OF $n \cdot \log_{10} 2$ IS UNIFORM MEASURE.

IN TABULATED DATA, CENSUS NUMBERS OR OTHER LARGE STAT DATABASES,

THE FRACTION OF NUMBERS W/ LEADING DIGIT a

$\sim \log_{10} (a+1) - \log_{10} a = \log_{10} \frac{a+1}{a}$.

1 IS MOST COMMON, THEN 2, 3, ...

THIS IS CALLED "BENFORD'S LAW".

#20: IF $x^2 = x$

THEN $(1-x)^2 = 1 - 2x + x^2$

$= 1 - x.$

PAIR $x, 1-x$, AN INVOLUTION.