Brownian motion and random walk

This lecture follows Mörters and Peres, Chapter 5.
Law of the iterated logarithm

**Theorem**

Suppose \( \{B(t) : t \geq 0\} \) is a standard linear Brownian motion. Then, almost surely,

\[
\limsup_{t \to \infty} \frac{B(t)}{\sqrt{2t \log \log t}} = 1.
\]
Proof.

- Fix $\epsilon > 0$ and $q > 1$. Let $\psi(t) = \sqrt{2t \log \log t}$ and

\[ A_n = \left\{ \max_{0 \leq t \leq q^n} B(t) \geq (1 + \epsilon)\psi(q^n) \right\}. \]

- Since the distribution of the maximum up to time $t$ is the same as for $|B(t)|$,

\[ \text{Prob}(A_n) = \text{Prob}\left\{ \frac{|B(q^n)|}{\sqrt{q^n}} \geq (1 + \epsilon)\frac{\psi(q^n)}{\sqrt{q^n}} \right\}. \]

- For $Z$ standard normal, $\text{Prob}(Z > x) \leq e^{-x^2/2}$, so

\[ \text{Prob}(A_n) \leq 2 \exp\left(-(1 + \epsilon)^2 \log \log q^n\right) = \frac{2}{(n \log q)^{(1+\epsilon)^2}}. \]
Law of the iterated logarithm

Proof.

- Since the bound is summable in $n$ we get that, almost surely, $A_n$ occurs only finitely often.
- For large $t$, $q^{n-1} \leq t < q^n$, we have

$$\frac{B(t)}{\psi(t)} = \frac{B(t)}{\psi(q^n)} \frac{\psi(q^n)}{q^n} \frac{t}{\psi(t)} \frac{q^n}{q^{n-1}} \leq (1 + \epsilon)q,$$

so that

$$\limsup \frac{B(t)}{\psi(t)} \leq (1 + \epsilon)q, \text{ a.s.}$$

Letting $\epsilon \downarrow 0$ and $q \downarrow 1$ we get the upper bound.
Proof.

- For the lower bound, let $q > 1$.
- Let
  \[ D_n = \{ B(q^n) - B(q^{n-1}) \geq \psi(q^n - q^{n-1}) \} . \]
- For a standard normal, there is $c > 0$ such that, for large $x$,
  \[ \text{Prob}(Z > x) \geq \frac{ce^{-x^2/2}}{x}. \]
  Thus
  \[
  \text{Prob}(D_n) \geq \text{Prob} \left( Z \geq \frac{\psi(q^n - q^{n-1})}{\sqrt{q^n - q^{n-1}}} \right) \geq c \frac{e^{-\log \log(q^n - q^{n-1})}}{\sqrt{2 \log \log(q^n - q^{n-1})}} \\
  \geq \frac{ce^{-\log(n \log q)}}{\sqrt{2 \log(n \log q)}} > \frac{\tilde{c}}{n \log n}.
  \]
  Since $\sum \text{Prob}(D_n) = \infty$, $D_n$ occurs i.o. almost surely.
Law of the iterated logarithm

Proof.

- Using the upper bound for $-B(q^{n-1})$, a.s. i.o.

$$\frac{B(q^n)}{\psi(q^n)} \geq \frac{-2\psi(q^{n-1}) + \psi(q^n - q^{n-1})}{\psi(q^n)}$$

$$\geq \frac{-2}{\sqrt{q}} + \frac{q^n - q^{n-1}}{q^n} = 1 - \frac{2}{\sqrt{q}} - \frac{1}{q}.$$

- Letting $q \uparrow \infty$ concludes the proof.
Suppose \( \{B(t) : t \geq 0\} \) is a standard Brownian motion. Then a.s.

\[
\limsup_{h \downarrow 0} \frac{|B(h)|}{\sqrt{2h \log \log (1/h)}} = 1.
\]

Proof.
This follows on using the time inversion \( X(t) = tB(1/t) \).
Lemma

If \( \{T_n : n \geq 1\} \) is a sequence of random times (not necessarily stopping times) satisfying \( T_n \to \infty \) and \( \frac{T_{n+1}}{T_n} \to 1 \) a.s., then

\[
\limsup_{n \to \infty} \frac{B(T_n)}{\psi(T_n)} = 1 \text{ a.s.}
\]

Also, if \( \frac{T_n}{n} \to a > 0 \) a.s. then

\[
\limsup_{n \to \infty} \frac{B(T_n)}{\psi(an)} = 1 \text{ a.s.}
\]
Law of the iterated logarithm

Proof.

- The upper bound follows from the previous theorem.
- Define, for $q > 4$,

$$
D_k = \{ B(q^k) - B(q^{k-1}) \geq \psi(q^k - q^{k-1}) \}
$$

$$
\Omega_k = \left\{ \min_{q^k \leq t \leq q^{k+1}} B(t) - B(q^k) \geq -\sqrt{q^k} \right\}, \quad D_k^* = D_k \cap \Omega_k.
$$

- Note $D_k$ and $\Omega_k$ are independent.

$$
\text{Prob}(D_k) = \text{Prob}\left\{ B(1) \geq \frac{\psi(q^k - q^{k-1})}{\sqrt{q^k - q^{k-1}}} \right\} \geq \frac{c}{k \log k}.
$$

Also $\text{Prob}(\Omega_k) =: c_q > 0$. 
Law of the iterated logarithm

**Proof.**

- The events \( \{D_{2k}^*: k \geq 1\} \) are independent and \( \sum_k \text{Prob}(D_{2k}^*) = \infty \), so they occur i.o. a.s., so that
  \[
  \min_{q^k \leq t \leq q^{k+1}} B(t) \geq \psi(q^k - q^{k-1}) - 2\psi(q^{k-1}) - \sqrt{q^k}.
  \]
  i.o., a.s. As \( q \uparrow \infty \), the RHS is \( \psi(q^k)(1 + o(1)) \).

- Now define \( n(k) = \min\{n : T_n > q^k\} \). Since \( T_{n+1}/T_n \to 1 \), it follows that \( q^k \leq T_{n(k)} < q^k(1 + \epsilon) \) for all large \( k \), so that
  \[
  \limsup_{n \to \infty} \frac{B(T_n)}{\psi(T_n)} \geq 1.
  \]
Law of the iterated logarithm

Theorem

Let \( \{S_n : n \geq 0\} \) be a simple random walk. Then, almost surely,

\[
\limsup_{n \to \infty} \frac{S_n}{\sqrt{2n \log \log n}} = 1.
\]
Proof.

- Let $T_0 = 0$, and, for $n \geq 1$,
  \[ T_n = \min(t > T_{n-1} : |B(t) - B(T_{n-1})| = 1). \]

- Evidently, $B(T_n)$ is simple random walk.

- The waiting times $T_n - T_{n-1}$ are i.i.d. and $E[T_n - T_{n-1}] = 1$ so the l.l.n. implies $\frac{T_n}{n} \to 1$ a.s., which reduces simple random walk to the previous theorem.
Theorem (Skorokhod embedding theorem)

Let \( \{B(t) : t \geq 0\} \) be a standard Brownian motion and let \( X \) be a real random variable with \( \mathbb{E}[X] = 0 \) and \( \mathbb{E}[X^2] < \infty \). Then there exists a stopping time \( T \), with respect to the natural filtration \( (\mathcal{F}(t) : t \geq 0) \) of the Brownian motion, such that \( B(T) \) has the law of \( X \) and \( \mathbb{E}[T] = \mathbb{E}[X^2] \).
Combining the Skorokhod embedding theorem with the argument giving the law of the iterated logarithm for simple random walk obtains the following more general version.

**Theorem (Hartman-Wintner law of the iterated logarithm)**

Let \( \{S_n : n \in \mathbb{N}\} \) be a random walk with increments \( S_n - S_{n-1} \) of zero mean and finite variance \( \sigma^2 \). Then

\[
\limsup_{n \to \infty} \frac{S_n}{\sqrt{2\sigma^2 n \log \log n}} = 1.
\]
Dubins’ embedding theorem

We say that a martingale \( \{X_n : n \in \mathbb{N}\} \) is a \textit{binary splitting} if, whenever for some \( x_0, x_1, ..., x_n \in \mathbb{R} \) the event

\[
A(x_0, ..., x_n) := \{ X_0 = x_0, X_1 = x_1, ..., X_n = x_n \}
\]

has positive probability, the random variable \( X_{n+1} \) conditioned on \( A(x_0, ..., x_n) \) takes on at most two values.
Dubins’ embedding theorem

Lemma

Let $X$ be a random variable with $E[X^2] < \infty$. Then there is a binary splitting martingale $\{X_n : n \in \mathbb{N}\}$ such that $X_n \to X$ a.s. and in $L^2$. 
Proof.

Let \( X_0 = E[X] \). Define, iteratively,

\[
\xi_n = \begin{cases} 
1 & X \geq X_n \\
-1 & X < X_n 
\end{cases}
\]

\[ G_n = \sigma(\xi_0, \xi_1, \ldots, \xi_{n-1}) \]

\[ X_n = E[X|G_n]. \]

So defined, \( X_n \) is a binary splitting martingale. Also,

\[
E[X^2] = E[(X - X_n)^2] + E[X_n^2] \geq E[X_n^2].
\]
**Dubins’ embedding theorem**

**Proof.**

- Since \( \{X_n\} \) is bounded in \( L^2 \), it follows that
  
  \[
  X_n \to X_\infty := \mathbb{E}[X|\mathcal{G}_\infty],
  \]

  a.s. and in \( L^2 \), where \( \mathcal{G}_\infty = \sigma(\bigcup_{i=0}^{\infty} \mathcal{G}_i) \).

- We claim
  
  \[
  \lim_{n \uparrow \infty} \xi_n(X - X_{n+1}) = |X - X_\infty|.
  \]

  This holds where \( X(\omega) = X_\infty(\omega) \). If \( X(\omega) < X_\infty(\omega) \) then \( X_n(\omega) > X(\omega) \) for all \( n \) sufficiently large, so that, for these \( n \), \( \xi_n(\omega) = -1 \) and the claim holds. The case \( X(\omega) > X_\infty(\omega) \) is similar.
Dubins’ embedding theorem

Proof.

We have

$$E[\xi_n (X - X_{n+1})] = E[\xi_n E[X - X_{n+1} | \mathcal{G}_{n+1}]] = 0.$$  

Since $\xi_n (X - X_{n+1})$ is bounded in $L^2$, $E[|X - X_\infty|] = 0$. 

$\square$
Proof of the Skorokhod embedding theorem.

- Let \( \{X_n : n \in \mathbb{N}\} \) be a binary splitting martingale \( X_n \to X \) a.s. and in \( L^2 \).

- Choose a sequence of stopping times \( T_0 \leq T_1 \leq \ldots \) such that \( B(T_n) \) is distributed as \( X_n \) and \( E[T_n] = E[X_n^2] \).

- As \( T_n \) is an increasing sequence, we have \( T_n \uparrow T \) a.s. for some stopping time \( T \). Moreover,

\[
E[T] = \lim_{n \uparrow \infty} E[T_n] = \lim_{n \uparrow \infty} E[X_n^2] = E[X^2].
\]

- Since \( B(T_n) \) converges in distribution to \( X \), and converges a.s. to \( B(T) \) by continuity, we have \( B(T) \) is distributed as \( X \).
The Donsker invariance principle

Let \( \{X_n : n \geq 0\} \) be a sequence of i.i.d. random variables with \( \mathbb{E}[X_n] = 0 \) and \( \mathbb{V}(X_n) = 1 \). Let

\[
S_n = \sum_{k=1}^{n} X_k.
\]

Define

\[
S(t) = S_{\lfloor t \rfloor} + (t - \lfloor t \rfloor)(S_{\lfloor t \rfloor + 1} - S_{\lfloor t \rfloor}).
\]

Define a sequence \( \{S_n^* : n \geq 1\} \) of random functions in \( C[0, 1] \) by

\[
S_n^*(t) = \frac{S(nt)}{\sqrt{n}}, \quad t \in [0, 1].
\]
The Donsker invariance principle

**Theorem (Donsker’s invariance principle)**

*On the space $C[0, 1]$ of continuous functions on the unit interval with sup norm, the sequence $\left\{ S_n^* : n \geq 1 \right\}$ converges in distribution to a standard Brownian motion $\{ B(t) : t \in [0, 1] \}$.***

This theorem is also known as the functional central limit theorem.
The Donsker invariance principle

**Lemma**

Suppose \( \{B(t) : t \geq 0\} \) is a linear Brownian motion. Then, for any random variable \( X \) with mean 0 and variance 1, there exists a sequence of stopping times

\[
0 = T_0 \leq T_1 \leq T_2 \leq T_3 \leq ... 
\]

with respect to the Brownian motion, such that

1. The sequence \( \{B(T_n) : n \geq 0\} \) has the distribution of the random walk with increments given by the law of \( X \)
2. The sequence of functions \( \{S_n^* : n \geq 0\} \) constructed from this random walk satisfies

\[
\lim_{n \to \infty} \text{Prob} \left( \sup_{0 \leq t \leq 1} \left| \frac{B(nt)}{\sqrt{n}} - S_n^*(t) \right| > \epsilon \right) = 0.
\]
Proof.

- Let $T_1$ be a stopping time with $\mathbb{E}[T_1] = 1$, such that $B(T_1) = X$ in distribution.
- By the strong Markov property,

$$\{B_2(t) : t \geq 0\} = \{B(T_1 + t) - B(T_1) : t \geq 0\}$$

is a Brownian motion independent of $\mathcal{F}^+(T_1)$.

- It follows that there is a sequence of stopping times $0 = T_0 \leq T_1 \leq T_2 \leq \ldots$ such that $S_n = B(T_n)$ is the embedded random walk and $\mathbb{E}[T_n] = n$. 
The Donsker invariance principle

Proof.

- Define $W_n(t) = \frac{B(nt)}{\sqrt{n}}$ and let $A_n$ be the event that there exists $t \in [0, 1)$ such that $|S_n^*(t) - W_n(t)| > \epsilon$.
- Let $k = k(t)$ be the unique integer with $\frac{k-1}{n} \leq t < \frac{k}{n}$. Since $S_n^*$ linearly interpolates values

$$
A_n \subset \{ \exists \ t \in [0, 1), |S_k/\sqrt{n} - W_n(t)| > \epsilon \} \\
\cup \{ \exists \ t \in [0, 1), |S_{k-1}/\sqrt{n} - W_n(t)| > \epsilon \}.
$$

- Recall $S_k = B(T_k) = \sqrt{n}W_n(T_k/n)$. For $0 < \delta < 1$, $A_n$ is contained in

$$
\{ \exists \ s, t \in [0, 2], \text{ s.t. } |s - t| < \delta, |W_n(s) - W_n(t)| > \epsilon \} \\
\cup \{ \exists \ t \in [0, 1), \text{ s.t. } |T_k/n - t| \vee |T_{k-1}/n - t| \geq \delta \}.
$$
The Donsker invariance principle

Proof.

- Since Brownian motion is uniformly continuous on $[0, 2]$, the first item may be made arbitrarily small in probability by choosing $\delta$ sufficiently small.

- To bound the second set for fixed $\delta$, note that

$$
\lim_{n \to \infty} \frac{T_n}{n} = \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} (T_k - T_{k-1}) = 1 \text{ a.s.}
$$

- Now check $\sum_{k=1}^{\alpha n} (T_k - T_{k-1})$ at rationals $\alpha = \frac{a}{M}$, $0 \leq a \leq M$ for sufficiently large $M$, and use that the sum is increasing in $\alpha$. 
Proof of the Donsker invariance principle.

- Choose stopping times as in the proof of the previous lemma, and recall that \( W_n(t) = \frac{B(nt)}{\sqrt{n}} \) is a standard Brownian motion.

- Suppose \( K \subset C[0, 1] \) is closed and define

\[
K[\epsilon] = \{ f \in C[0, 1] : \| f - g \|_\infty \leq \epsilon, \text{ some } g \in K \}.
\]

- Bound

\[
\text{Prob}(S_n^* \in K) \leq \text{Prob}(W_n \in K[\epsilon]) + \text{Prob}(\| S_n^* - W_n \|_\infty > \epsilon).
\]

The second term tends to 0 as \( n \to \infty \).
Proof of the Donsker invariance principle.

- The first term is equal to $\text{Prob}(B \in K[\epsilon])$. Since

  $$\lim_{\epsilon \downarrow 0} \text{Prob}(B \in K[\epsilon]) = \text{Prob}(B \in K),$$

  $$\limsup_{n \to \infty} \text{Prob}(S_n^* \in K) \leq \text{Prob}(B \in K),$$

  which suffices to prove the convergence in distribution.
As an example of the functional CLT we prove the following limit theorem.

**Theorem**

Suppose that \( \{X_k : k \geq 1\} \) is a sequence of i.i.d. random variables with \( \mathbb{E}[X_1] = 0 \) and \( \mathbb{E}[X_1^2] = 1 \). Let \( \{S_n : n \geq 0\} \) be the associated random walk and

\[
M_n = \max\{S_k : 0 \leq k \leq n\}.
\]

For all \( x \geq 0 \),

\[
\lim_{n \to \infty} \operatorname{Prob}(M_n \geq x\sqrt{n}) = \frac{2}{\sqrt{2\pi}} \int_x^\infty e^{-y^2/2} \, dy.
\]
The Donsker invariance principle

Proof.

- Let $g : \mathbb{R} \to \mathbb{R}$ be a bounded continuous function.

- Define $G : C[0, 1] \to \mathbb{R}$ by

$$G(f) = g \left( \max_{x \in [0, 1]} f(x) \right).$$

This is continuous and bounded.

- We have

$$E[G(S_n^*)] = E \left[ g \left( \frac{\max_{0 \leq k \leq n} S_k}{\sqrt{n}} \right) \right], \quad E[G(B)] = E \left[ g \left( \max_{0 \leq t \leq 1} B(t) \right) \right].$$
The Donsker invariance principle

**Proof.**

- By the functional CLT,
  \[
  \lim_{n \to \infty} E \left[ g \left( \frac{M_n}{\sqrt{n}} \right) \right] = E \left[ g \left( \max_{0 \leq t \leq 1} B(t) \right) \right].
  \]

- Hence, by the reflection principle
  \[
  \lim_{n \to \infty} \text{Prob}(M_n \geq x\sqrt{n}) = 2 \text{Prob}(|B(1)| \geq x).
  \]
The arcsine distribution is the distribution on \((0, 1)\) with density

\[
\frac{1}{\pi \sqrt{x(1-x)}}.
\]

The cumulative distribution function of a variable \(X\) with arcsine distribution is given by

\[
\text{Prob}(X \leq x) = \frac{2}{\pi} \arcsin(\sqrt{x}), \quad x \in (0, 1).
\]
The arcsine laws

Theorem (First arcsine law)

Let \( \{B(t) : t \geq 0\} \) be a standard linear Brownian motion. Then

1. The random variable \( L = \sup \{t \in [0, 1] : B(t) = 0\} \) has an arcsine distribution

2. The location \( M^* \) of \( \max B(s) \) in \([0, 1]\) has an arcsine distribution.
The arcsine laws

Proof.

- Let $M(t) = \max_{0 \leq s \leq t} B(s)$. Since $M(t) - B(t)$ has the distribution of $|B(t)|$, the two distributions in the theorem are the same, and it suffices to prove the second claim.

- We have

$$\text{Prob}(M^* < s) = \text{Prob}\left( \max_{0 \leq u \leq s} B(u) > \max_{s \leq v \leq 1} B(v) \right)$$

$$= \text{Prob}\left( \max_{0 \leq u \leq s} B(u) - B(s) > \max_{s \leq v \leq 1} B(v) - B(s) \right)$$

$$= \text{Prob}(M_1(s) > M_2(1 - s))$$

where $M_1$ and $M_2$ are independent maximum processes of Brownian motion.
Proof.

- We have, for independent standard normals $Z_1, Z_2$,

$$\text{Prob}(M_1(s) > M_2(1 - s)) = \text{Prob}(|B_1(s)| > |B_2(1 - s)|)$$

$$= \text{Prob}(\sqrt{s}|Z_1| > \sqrt{1 - s}|Z_2|)$$

$$= \text{Prob} \left( \frac{|Z_2|}{\sqrt{Z_1^2 + Z_2^2}} < \sqrt{s} \right).$$

- Since the 2d Gaussian has spherical symmetry, this gives the arcsine law.
The arcsine laws

**Theorem**

Suppose that \( \{X_k : k \geq 1\} \) is a sequence of i.i.d. random variables with \( \mathbb{E}[X_1] = 0 \) and \( \text{Var}[X_1] = 1 \). Let \( \{S_n : n \geq 0\} \) be the associated random walk and

\[
N_n = \max\{1 \leq k \leq n : S_k S_{k-1} \leq 0\}.
\]

Then, for all \( x \in (0, 1) \),

\[
\lim_{n \to \infty} \text{Prob}(N_n \leq xn) = \frac{2}{\pi} \arcsin(\sqrt{x}).
\]
The arcsine laws

Proof.

- Define bounded function \( g \) on \( C[0, 1] \) by

\[
g(f) = \max(t \leq 1 : f(t) = 0)
\]

or 0 if no zero exists.

- We have that \( g(S_n^*) \) differs from \( \frac{N_n}{n} \) by an amount which is \( O(1/n) \).

- \( g \) is not continuous on \( C[0, 1] \) but it is continuous on the subset \( \mathcal{C} \) of functions \( f \) such that \( f(1) \neq 0 \) and such that \( f \) takes positive and negative values in every neighborhood of a zero. Note that \( B \in \mathcal{C} \) a.s.
The arcsine laws

Proof.

By Donsker’s invariance principle, for every bounded continuous function $h : \mathbb{R} \to \mathbb{R}$,

$$\lim_{n \to \infty} E \left[ h \left( \frac{N_n}{n} \right) \right] = \lim_{n \to \infty} E[h \circ g(S_n^*)] = E[h \circ g(B)]$$

so that the claim follows from the previous theorem.
Theorem

Let \( \{B(t) : t \geq 0\} \) be a standard Brownian motion. Then \( \text{meas}(t \in [0, 1] : B(t) > 0) \) is arcsine distributed.
The second arcsine law

Lemma

Let $S_k$ be a simple symmetric random walk on the integers. Then

$\# \{ k \in \{1, ..., n\} : S_k > 0 \}$ is equal in distribution to

$\min\{ k \in \{0, ..., n\} : S_k = \max_{0 \leq j \leq n} S_j \}$.

The proof is a bijection, see MP pp. 138-139.
Proof of the second arcsine law.

- Define

\[ g(f) = \inf\{ t \in [0, 1] : f(t) = \sup_{s \in [0,1]} f(s) \} \]

This is continuous on the set of \( f \in C[0, 1] \) having a unique maximum, which contains Brownian motion a.s.

- By the Donsker invariant theorem

\[
\frac{1}{n} \min \left\{ k \in \{0, \ldots, n\} : S_k = \max_{0 \leq j \leq n} S_j \right\}
\]

converges in distribution to \( g(B) \), which has an arcsine distribution.
The second arcsine law

Proof of the second arcsine law.

Let

\[ h(f) = \text{meas}\{t \in [0, 1] : f(t) > 0\}. \]

Then

\[ \frac{1}{n}\#\{k \in \{1, \ldots, n\} : S_k > 0\} \]

is approximated by \(h(S_n^*)\) in probability.

\(h\) is continuous on the set of \(f \in C[0, 1]\) satisfying

\[ \lim_{\epsilon \downarrow 0} \text{meas}(t \in [0, 1] : |f(t)| \leq \epsilon) = 0 \]

which holds for Brownian motion a.s. Thus, applying Donsker again, one obtains the arcsine law.
Theorem

Let \( \{B(t) : t \geq 0\} \) be a standard linear Brownian motion and, for \( a \geq 0 \), let \( \tau_a = \inf\{t \geq 0 : B(t) = a\} \) and \( \sigma_a = \inf\{t \geq 0 : |B(t)| = a\} \). Then

\[
\int_0^{\tau_a} 1(0 \leq B(t) \leq a) \, dt \overset{d}{=} \sigma_a.
\]
Lemma

Let \( s(t) = \int_0^t 1(B(s) \geq 0) \, ds \) and let \( t(s) = \inf\{ t \geq 0 : s(t) \geq s \} \) its right-continuous inverse. Then

\[ \{ B(t(s)) : s \geq 0 \} \overset{d}{=} \{ |B(s)| : s \geq 0 \}. \]
Proof.

Let \( \{S(n) : n = 0, 1, \ldots\} \) be simple random walk, and let \( \{S_n^*(s) : s \geq 0\} \) be defined by linear interpolation as in the functional central limit theorem.

Define

\[
s(t, f) = \int_0^t 1(f(s) \geq 0) \, ds, \quad t(s, f) = \inf(t \geq 0 : s(t, f) \geq s)
\]

Removing the negative excursions from simple random walk gives reflected random walk, so

\[
\{S_n^*(t(s, S_n^*)) : s \geq 0\} \overset{d}{=} \{|S_n^*(s)| : s \geq 0\}.
\]
Proof.

Since the mapping $f \mapsto f(t(\cdot, f))$ is continuous on the part of $C[0, 1]$ for which

$$\lim_{\epsilon \downarrow 0} \operatorname{meas}\{s \in [0, t]: -\epsilon \leq f(s) \leq \epsilon\} = 0$$

which holds for Brownian motion with probability 1, the equality in distribution for Brownian motion holds by the functional central limit theorem.
Proof of theorem.

Observe

$$\int_{0}^{T_{a}} 1(0 \leq B(s) \leq a) ds = \inf \{s \geq 0 : B(t(s)) = a\}.$$ 

Also,

$$\inf \{s \geq 0 : B(t(s)) = a\} \overset{d}{=} \inf \{s \geq 0 : |B(s)| = a\} = \sigma_{a}.$$