Math 639: Lecture 21

Brownian motion and random walk

Bob Hough

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Brownian motion and random walk

This lecture follows Mörters and Peres, Chapter 5.

Law of the iterated logarithm

Theorem

Suppose $\{B(t) : t \geqslant 0\}$ is a standard linear Brownian motion. Then, almost surely,

$$\limsup_{t\to\infty} \frac{B(t)}{\sqrt{2t\log\log t}} = 1.$$

Law of the iterated logarithm

Proof.

- Fix $\epsilon > 0$ and $q > 1$. Let $\psi(t) = \sqrt{2t \log\log t}$ and

$$A_n = \left\{ \max_{0 \leqslant t \leqslant q^n} B(t) \geqslant (1+\epsilon)\psi(q^n) \right\}.$$

- Since the distribution of the maximum up to time t is the same as for $|B(t)|$,

$$\mathrm{Prob}(A_n) = \mathrm{Prob}\left\{ \frac{|B(q^n)|}{\sqrt{q^n}} \geqslant (1+\epsilon)\frac{\psi(q^n)}{\sqrt{q^n}} \right\}.$$

- For Z standard normal, $\mathrm{Prob}(Z > x) \leqslant e^{-x^2/2}$, so

$$\mathrm{Prob}(A_n) \leqslant 2\exp\left(-(1+\epsilon)^2 \log\log q^n\right) = \frac{2}{(n \log q)^{(1+\epsilon)^2}}.$$

$\square$

Law of the iterated logarithm

Proof.

- Since the bound is summable in n we get that, almost surely, A_n occurs only finitely often.
- For large t, $q^{n-1} \leqslant t < q^n$, we have

$$\frac{B(t)}{\psi(t)} = \frac{B(t)}{\psi(q^n)} \frac{\psi(q^n)}{q^n} \frac{t}{\psi(t)} \frac{q^n}{t} \leqslant (1+\epsilon)q,$$

so that

$$\limsup \frac{B(t)}{\psi(t)} \leqslant (1+\epsilon)q, \; a.s.$$

Letting $\epsilon \downarrow 0$ and $q \downarrow 1$ we get the upper bound.

$\square$

Law of the iterated logarithm

Proof.

- For the lower bound, let $q > 1$.
- Let
$$D_n = \left\{B(q^n) - B(q^{n-1}) \geqslant \psi(q^n - q^{n-1})\right\}.$$
- For a standard normal, there is $c > 0$ such that, for large x, $\mathrm{Prob}(Z > x) \geqslant \frac{ce^{-x^2/2}}{x}$. Thus

$$\mathrm{Prob}(D_n) \geqslant \mathrm{Prob}\left(Z \geqslant \frac{\psi(q^n - q^{n-1})}{\sqrt{q^n - q^{n-1}}}\right) \geqslant c\frac{e^{-\log\log(q^n - q^{n-1})}}{\sqrt{2\log\log(q^n - q^{n-1})}}$$
$$\geqslant \frac{ce^{-\log(n\log q)}}{\sqrt{2\log(n\log q)}} > \frac{\tilde{c}}{n\log n}.$$

Since $\sum \mathrm{Prob}(D_n) = \infty$, D_n occurs i.o. almost surely.

$\square$

Law of the iterated logarithm

Proof.

- Using the upper bound for $-B(q^{n-1})$, a.s. i.o.

$$\frac{B(q^n)}{\psi(q^n)} \geqslant \frac{-2\psi(q^{n-1}) + \psi(q^n - q^{n-1})}{\psi(q^n)}$$
$$\geqslant \frac{-2}{\sqrt{q}} + \frac{q^n - q^{n-1}}{q^n} = 1 - \frac{2}{\sqrt{q}} - \frac{1}{q}.$$

- Letting $q \uparrow \infty$ concludes the proof.

$\square$

Law of the iterated logarithm

Corollary

Suppose $\{B(t) : t \geqslant 0\}$ *is a standard Brownian motion. Then a.s.*

$$\limsup_{h \downarrow 0} \frac{|B(h)|}{\sqrt{2h \log \log(1/h)}} = 1.$$

Proof.

This follows on using the time inversion $X(t) = tB(1/t)$. $\square$

Law of the iterated logarithm

Lemma

If $\{T_n : n \geqslant 1\}$ is a sequence of random times (not necessarily stopping times) satisfying $T_n \to \infty$ and $\frac{T_{n+1}}{T_n} \to 1$ a.s., then

$$\limsup_{n\to\infty} \frac{B(T_n)}{\psi(T_n)} = 1 \text{ a.s.}$$

Also, if $\frac{T_n}{n} \to a > 0$ a.s. then

$$\limsup_{n\to\infty} \frac{B(T_n)}{\psi(an)} = 1 \text{ a.s.}$$

Law of the iterated logarithm

Proof.

- The upper bound follows from the previous theorem.
- Define, for $q > 4$,

$$D_k = \{B(q^k) - B(q^{k-1}) \geqslant \psi(q^k - q^{k-1})\}$$

$$\Omega_k = \left\{\min_{q^k \leqslant t \leqslant q^{k+1}} B(t) - B(q^k) \geqslant -\sqrt{q^k}\right\}, \; D_k^* = D_k \cap \Omega_k.$$

- Note D_k and Ω_k are independent.

$$\mathrm{Prob}(D_k) = \mathrm{Prob}\left\{B(1) \geqslant \frac{\psi(q^k - q^{k-1})}{\sqrt{q^k - q^{k-1}}}\right\} \geqslant \frac{c}{k \log k}.$$

Also $\mathrm{Prob}(\Omega_k) =: c_q > 0$.

□

Law of the iterated logarithm

Proof.

- The events $\{D_{2k}^* : k \geqslant 1\}$ are independent and $\sum_k \text{Prob}(D_{2k}^*) = \infty$, so they occur i.o. a.s., so that

$$\min_{q^k \leqslant t \leqslant q^{k+1}} B(t) \geqslant \psi(q^k - q^{k-1}) - 2\psi(q^{k-1}) - \sqrt{q^k}.$$

  i.o., a.s. As $q \uparrow \infty$, the RHS is $\psi(q^k)(1 + o(1))$.

- Now define $n(k) = \min\{n : T_n > q^k\}$. Since $T_{n+1}/T_n \to 1$, it follows that $q^k \leqslant T_{n(k)} < q^k(1+\epsilon)$ for all large k, so that

$$\limsup_{n \to \infty} \frac{B(T_n)}{\psi(T_n)} \geqslant 1.$$

$\square$

Law of the iterated logarithm

Theorem

Let $\{S_n : n \geqslant 0\}$ be a simple random walk. Then, almost surely,

$$\limsup_{n\to\infty} \frac{S_n}{\sqrt{2n\log\log n}} = 1.$$

Law of the iterated logarithm

Proof.

- Let $T_0 = 0$, and, for $n \geqslant 1$,

$$T_n = \min(t > T_{n-1} : |B(t) - B(T_{n-1})| = 1).$$

- Evidently, $B(T_n)$ is simple random walk.
- The waiting times $T_n - T_{n-1}$ are i.i.d. and $\mathrm{E}[T_n - T_{n-1}] = 1$ so the l.l.n. implies $\frac{T_n}{n} \to 1$ a.s., which reduces simple random walk to the previous theorem.

$\square$

Skorokhod embedding theorem

Theorem (Skorokhod embedding theorem)

Let $\{B(t) : t \geqslant 0\}$ *be a standard Brownian motion and let* X *be a real random variable with* $\mathrm{E}[X] = 0$ *and* $\mathrm{E}[X^2] < \infty$*. Then there exists a stopping time* T*, with respect to the natural filtration* $(\mathscr{F}(t) : t \geqslant 0)$ *of the Brownian motion, such that* $B(T)$ *has the law of* X *and* $\mathrm{E}[T] = \mathrm{E}[X^2]$*.*

Hartman-Wintner law of the iterated logarithm

Combining the Skorokhod embedding theorem with the argument giving the law of the iterated logarithm for simple random walk obtains the following more general version.

Theorem (Hartman-Wintner law of the iterated logarithm)

Let $\{S_n : n \in \mathbb{N}\}$ be a random walk with increments $S_n - S_{n-1}$ of zero mean and finite variance σ^2. Then

$$\limsup_{n\to\infty} \frac{S_n}{\sqrt{2\sigma^2 n \log\log n}} = 1.$$

Dubins' embedding theorem

We say that a martingale $\{X_n : n \in \mathbb{N}\}$ is a *binary splitting* if, whenever for some $x_0, x_1, ..., x_n \in \mathbb{R}$ the event

$$A(x_0, ..., x_n) := \{X_0 = x_0, X_1 = x_1, ..., X_n = x_n\}$$

has positive probability, the random variable X_{n+1} conditioned on $A(x_0, ..., x_n)$ takes on at most two values.

Dubins' embedding theorem

Lemma

Let X be a random variable with $\mathrm{E}[X^2] < \infty$. Then there is a binary splitting martingale $\{X_n : n \in \mathbb{N}\}$ such that $X_n \to X$ a.s. and in L^2.

Dubins' embedding theorem

Proof.

- Let $X_0 = \mathrm{E}[X]$. Define, iteratively,

$$\xi_n = \begin{cases} 1 & X \geqslant X_n \\ -1 & X < X_n \end{cases}$$
$$\mathscr{G}_n = \sigma(\xi_0, \xi_1, \ldots, \xi_{n-1})$$
$$X_n = \mathrm{E}[X|\mathscr{G}_n].$$

So defined, X_n is a binary splitting martingale. Also,

$$\mathrm{E}[X^2] = \mathrm{E}[(X - X_n)^2] + \mathrm{E}[X_n^2] \geqslant \mathrm{E}[X_n^2].$$

$\square$

Proof.

- Since $\{X_n\}$ is bounded in L^2, it follows that

$$X_n \to X_\infty := \mathrm{E}[X|\mathscr{G}_\infty],$$

  a.s. and in L^2, where $\mathscr{G}_\infty = \sigma\left(\bigcup_{i=0}^{\infty} \mathscr{G}_i\right)$.
- We claim

$$\lim_{n\uparrow\infty} \xi_n(X - X_{n+1}) = |X - X_\infty|.$$

  This holds where $X(\omega) = X_\infty(\omega)$. If $X(\omega) < X_\infty(w)$ then $X_n(\omega) > X(\omega)$ for all n sufficiently large, so that, for these n, $\xi_n(\omega) = -1$ and the claim holds. The case $X(\omega) > X_\infty(\omega)$ is similar.

$\square$

Dubins' embedding theorem

Proof.

- We have

$$\mathrm{E}[\xi_n(X - X_{n+1})] = \mathrm{E}[\xi_n \mathrm{E}[X - X_{n+1}|\mathscr{G}_{n+1}]] = 0.$$

- Since $\xi_n(X - X_{n+1})$ is bounded in L^2, $\mathrm{E}[|X - X_\infty|] = 0$.

$\square$

Skorokhod embedding theorem

Proof of the Skorokhod embedding theorem.

- Let $\{X_n : n \in \mathbb{N}\}$ be a binary splitting martingale $X_n \to X$ a.s. and in L^2.
- Choose a sequence of stopping times $T_0 \leqslant T_1 \leqslant \ldots$ such that $B(T_n)$ is distributed as X_n and $\mathrm{E}[T_n] = \mathrm{E}[X_n^2]$.
- As T_n is an increasing sequence, we have $T_n \uparrow T$ a.s. for some stopping time T. Moreover,

$$\mathrm{E}[T] = \lim_{n\uparrow\infty} \mathrm{E}[T_n] = \lim_{n\uparrow\infty} \mathrm{E}[X_n^2] = \mathrm{E}[X^2].$$

- Since $B(T_n)$ converges in distribution to X, and converges a.s. to $B(T)$ by continuity, we have $B(T)$ is distributed as X.

$\square$

The Donsker invariance principle

Let $\{X_n : n \geqslant 0\}$ be a sequence of i.i.d. random variables with $\mathrm{E}[X_n] = 0$ and $\mathrm{Var}(X_n) = 1$. Let

$$S_n = \sum_{k=1}^{n} X_k.$$

Define

$$S(t) = S_{[t]} + (t - [t])(S_{[t]+1} - S_{[t]}).$$

Define a sequence $\{S_n^* : n \geqslant 1\}$ of random functions in $C[0,1]$ by

$$S_n^*(t) = \frac{S(nt)}{\sqrt{n}}, \qquad t \in [0,1].$$

The Donsker invariance principle

Theorem (Donsker's invariance principle)

On the space $C[0,1]$ of continuous functions on the unit interval with sup norm, the sequence $\{S_n^* : n \geqslant 1\}$ converges in distribution to a standard Brownian motion $\{B(t) : t \in [0,1]\}$.

This theorem is also known as the functional central limit theorem.

The Donsker invariance principle

Lemma

Suppose $\{B(t) : t \geqslant 0\}$ is a linear Brownian motion. Then, for any random variable X with mean 0 and variance 1, there exists a sequence of stopping times

$$0 = T_0 \leqslant T_1 \leqslant T_2 \leqslant T_3 \leqslant \ldots$$

with respect to the Brownian motion, such that

1. *The sequence $\{B(T_n) : n \geqslant 0\}$ has the distribution of the random walk with increments given by the law of X*
2. *The sequence of functions $\{S_n^* : n \geqslant 0\}$ constructed from this random walk satisfies*

$$\lim_{n\to\infty} \operatorname{Prob}\left(\sup_{0\leqslant t\leqslant 1}\left|\frac{B(nt)}{\sqrt{n}} - S_n^*(t)\right| > \epsilon\right) = 0.$$

Proof.

- Let T_1 be a stopping time with $\mathrm{E}[T_1] = 1$, such that $B(T_1) = X$ in distribution.
- By the strong Markov property,

$$\{B_2(t) : t \geqslant 0\} = \{B(T_1 + t) - B(T_1) : t \geqslant 0\}$$

  is a Brownian motion independent of $\mathscr{F}^+(T_1)$.
- It follows that there is a sequence of stopping times $0 = T_0 \leqslant T_1 \leqslant T_2 \leqslant \ldots$ such that $S_n = B(T_n)$ is the embedded random walk and $\mathrm{E}[T_n] = n$.

$\square$

The Donsker invariance principle

Proof.

- Define $W_n(t) = \frac{B(nt)}{\sqrt{n}}$ and let A_n be the event that there exists $t \in [0,1)$ such that $|S_n^*(t) - W_n(t)| > \epsilon$.
- Let $k = k(t)$ be the unique integer with $\frac{k-1}{n} \leqslant t < \frac{k}{n}$. Since S_n^* linearly interpolates values

$$\begin{aligned} A_n \subset & \{\exists\, t \in [0,1), |S_k/\sqrt{n} - W_n(t)| > \epsilon\} \\ & \cup \{\exists\, t \in [0,1), |S_{k-1}/\sqrt{n} - W_n(t)| > \epsilon\}. \end{aligned}$$

- Recall $S_k = B(T_k) = \sqrt{n} W_n(T_k/n)$. For $0 < \delta < 1$, A_n is contained in

$$\begin{gathered} \{\exists\, s,t \in [0,2],\ \text{s.t. } |s-t| < \delta, |W_n(s) - W_n(t)| > \epsilon\} \\ \cup \{\exists\, t \in [0,1),\ \text{s.t. } |T_k/n - t| \vee |T_{k-1}/n - t| \geqslant \delta\}. \end{gathered}$$

$\square$

The Donsker invariance principle

Proof.

- Since Brownian motion is uniformly continuous on $[0,2]$, the first item may be made arbitrarily small in probability by choosing δ sufficiently small.
- To bound the second set for fixed δ, note that

$$\lim_{n\to\infty} \frac{T_n}{n} = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} (T_k - T_{k-1}) = 1 \ a.s.$$

- Now check $\sum_{k=1}^{\alpha n}(T_k - T_{k-1})$ at rationals $\alpha = \frac{a}{M}$, $0 \leqslant a \leqslant M$ for sufficiently large M, and use that the sum is increasing in α.

$\square$

Proof of the Donsker invariance principle.

- Choose stopping times as in the proof of the previous lemma, and recall that $W_n(t) = \frac{B(nt)}{\sqrt{n}}$ is a standard Brownian motion.
- Suppose $K \subset C[0,1]$ is closed and define

$$K[\epsilon] = \{f \in C[0,1] : \|f - g\|_\infty \leqslant \epsilon, \quad \text{some } g \in K\}.$$

- Bound

$$\mathrm{Prob}(S_n^* \in K) \leqslant \mathrm{Prob}(W_n \in K[\epsilon]) + \mathrm{Prob}(\|S_n^* - W_n\|_\infty > \epsilon).$$

The second term tends to 0 as $n \to \infty$.

□

The Donsker invariance principle

Proof of the Donsker invariance principle.

- The first term is equal to $\mathrm{Prob}(B \in K[\epsilon])$. Since

$$\lim_{\epsilon \downarrow 0} \mathrm{Prob}(B \in K[\epsilon]) = \mathrm{Prob}(B \in K),$$

$\limsup_{n\to\infty} \mathrm{Prob}(S_n^* \in K) \leqslant \mathrm{Prob}(B \in K)$, which suffices to prove the convergence in distribution.

$\square$

The Donsker invariance principle

As an example of the functional CLT we prove the following limit theorem.

Theorem

Suppose that $\{X_k : k \geqslant 1\}$ *is a sequence of i.i.d. random variables with* $\mathrm{E}[X_1] = 0$ *and* $\mathrm{E}[X_1^2] = 1$*. Let* $\{S_n : n \geqslant 0\}$ *be the associated random walk and*

$$M_n = \max\{S_k : 0 \leqslant k \leqslant n\}.$$

For all $x \geqslant 0$*,*

$$\lim_{n\to\infty} \mathrm{Prob}(M_n \geqslant x\sqrt{n}) = \frac{2}{\sqrt{2\pi}} \int_x^\infty e^{-y^2/2} dy.$$

The Donsker invariance principle

Proof.

- Let $g : \mathbb{R} \to \mathbb{R}$ be a bounded continuous function.
- Define $G : C[0,1] \to \mathbb{R}$ by

$$G(f) = g\left(\max_{x\in[0,1]} f(x)\right).$$

This is continuous and bounded.

- We have

$$\mathrm{E}[G(S_n^*)] = \mathrm{E}\left[g\left(\frac{\max_{0\leqslant k\leqslant n} S_k}{\sqrt{n}}\right)\right], \quad \mathrm{E}[G(B)] = \mathrm{E}\left[g\left(\max_{0\leqslant t\leqslant 1} B(t)\right)\right]$$

$\square$

The Donsker invariance principle

Proof.

- By the functional CLT,

$$\lim_{n\to\infty} \mathrm{E}\left[g\left(\frac{M_n}{\sqrt{n}}\right)\right] = \mathrm{E}\left[g\left(\max_{0\leqslant t\leqslant 1} B(t)\right)\right].$$

- Hence, by the reflection principle

$$\lim_{n\to\infty} \mathrm{Prob}(M_n \geqslant x\sqrt{n}) = 2\,\mathrm{Prob}(|B(1)| \geqslant x).$$

$\square$

The arcsine laws

The *arcsine distribution* is the distribution on $(0,1)$ with density

$$\frac{1}{\pi\sqrt{x(1-x)}}.$$

The cumulative distribution function of a variable X with arcsine distribution is given by

$$\mathrm{Prob}(X \leqslant x) = \frac{2}{\pi}\arcsin(\sqrt{x}), \qquad x \in (0,1).$$

The arcsine laws

Theorem (First arcsine law)

Let $\{B(t) : t \geqslant 0\}$ be a standard linear Brownian motion. Then

1. *The random variable $L = \sup\{t \in [0,1] : B(t) = 0\}$ has an arcsine distribution*
2. *The location M^* of $\max B(s)$ in $[0,1]$ has an arcsine distribution.*

The arcsine laws

Proof.

- Let $M(t) = \max_{0 \leqslant s \leqslant t} B(s)$. Since $M(t) - B(t)$ has the distribution of $|B(t)|$, the two distributions in the theorem are the same, and it suffices to prove the second claim.
- We have

$$\begin{aligned} \mathrm{Prob}(M^* < s) &= \mathrm{Prob}(\max_{0 \leqslant u \leqslant s} B(u) > \max_{s \leqslant v \leqslant 1} B(v)) \\ &= \mathrm{Prob}(\max_{0 \leqslant u \leqslant s} B(u) - B(s) > \max_{s \leqslant v \leqslant 1} B(v) - B(s)) \\ &= \mathrm{Prob}(M_1(s) > M_2(1-s)) \end{aligned}$$

where M_1 and M_2 are independent maximum processes of Brownian motion.

□

The arcsine laws

Proof.

- We have, for independent standard normals Z_1, Z_2,

$$\begin{aligned}\mathrm{Prob}(M_1(s) > M_2(1-s)) &= \mathrm{Prob}(|B_1(s)| > |B_2(1-s)|) \\ &= \mathrm{Prob}(\sqrt{s}|Z_1| > \sqrt{1-s}|Z_2|) \\ &= \mathrm{Prob}\left(\frac{|Z_2|}{\sqrt{Z_1^2 + Z_2^2}} < \sqrt{s}\right).\end{aligned}$$

- Since the 2d Gaussian has spherical symmetry, this gives the arcsine law.

$\square$

The arcsine laws

Theorem

Suppose that $\{X_k : k \geqslant 1\}$ *is a sequence of i.i.d. random variables with* $\mathrm{E}[X_1] = 0$ *and* $\mathrm{Var}[X_1] = 1$. *Let* $\{S_n : n \geqslant 0\}$ *be the associated random walk and*

$$N_n = \max\{1 \leqslant k \leqslant n : S_k S_{k-1} \leqslant 0\}.$$

Then, for all $x \in (0, 1)$,

$$\lim_{n\to\infty} \mathrm{Prob}(N_n \leqslant xn) = \frac{2}{\pi} \arcsin(\sqrt{x}).$$

Proof.

- Define bounded function g on $C[0,1]$ by
$$g(f) = \max(t \leqslant 1 : f(t) = 0)$$
or 0 if no zero exists.
- We have that $g(S_n^*)$ differs from $\frac{N_n}{n}$ by an amount which is $O(1/n)$.
- g is not continuous on $C[0,1]$ but it is continuous on the subset $\mathscr{C}$ of functions f such that $f(1) \neq 0$ and such that f takes positive and negative values in every neighborhood of a zero. Note that $B \in \mathscr{C}$ a.s.

$\square$

Proof.

- By Donsker's invariance principle, for every bounded continuous function $h : \mathbb{R} \to \mathbb{R}$,

$$\lim_{n\to\infty} \mathrm{E}\left[h\left(\frac{N_n}{n}\right)\right] = \lim_{n\to\infty} \mathrm{E}[h \circ g(S_n^*)] = \mathrm{E}[h \circ g(B)]$$

so that the claim follows from the previous theorem.

$\square$

The second arcsine law

Theorem

Let $\{B(t) : t \geqslant 0\}$ *be a standard Brownian motion. Then* $\mathrm{meas}(t \in [0,1] : B(t) > 0)$ *is arcsine distributed.*

The second arcsine law

Lemma

Let S_k be a simple symmetric random walk on the integers. Then $\#\{k \in \{1, \ldots, n\} : S_k > 0\}$ is equal in distribution to $\min\{k \in \{0, \ldots, n\} : S_k = \max_{0 \leqslant j \leqslant n} S_j\}$.

The proof is a bijection, see MP pp. 138-139.

The second arcsine law

Proof of the second arcsine law.

- Define
$$g(f) = \inf\{t \in [0,1] : f(t) = \sup_{s\in[0,1]} f(s)\}.$$
This is continuous on the set of $f \in C[0,1]$ having a unique maximum, which contains Brownian motion a.s.
- By the Donsker invariant theorem
$$\frac{1}{n}\min\left\{k \in \{0,\ldots,n\} : S_k = \max_{0\leqslant j\leqslant n} S_j\right\}$$
converges in distribution to $g(B)$, which has an arcsine distribution.

□

The second arcsine law

Proof of the second arcsine law.

- Let
$$h(f) = \mathrm{meas}\{t \in [0,1] : f(t) > 0\}.$$
Then
$$\frac{1}{n}\#\{k \in \{1, \ldots, n\} : S_k > 0\}$$
is approximated by $h(S_n^*)$ in probability.
- h is continuous on the set of $f \in C[0,1]$ satisfying
$$\lim_{\epsilon \downarrow 0} \mathrm{meas}(t \in [0,1] : |f(t)| \leqslant \epsilon) = 0$$
which holds for Brownian motion a.s. Thus, applying Donsker again, one obtains the arcsine law.

□

Excursions

Theorem

Let $\{B(t) : t \geqslant 0\}$ *be a standard linear Brownian motion and, for* $a \geqslant 0$, *let* $\tau_a = \inf\{t \geqslant 0 : B(t) = a\}$ *and* $\sigma_a = \inf\{t \geqslant 0 : |B(t)| = a\}$. *Then*

$$\int_0^{\tau_a} \mathbf{1}(0 \leqslant B(t) \leqslant a)dt \overset{d}{=} \sigma_a.$$

Excursions

Lemma

Let $s(t) = \int_0^t \mathbf{1}(B(s) \geqslant 0)ds$ *and let* $t(s) = \inf\{t \geqslant 0 : s(t) \geqslant s\}$ *its right-continuous inverse. Then*

$$\{B(t(s)) : s \geqslant 0\} \stackrel{d}{=} \{|B(s)| : s \geqslant 0\}.$$

Excursions

Proof.

- Let $\{S(n) : n = 0, 1, \ldots\}$ be simple random walk, and let $\{S_n^*(s) : s \geqslant 0\}$ be defined by linear interpolation as in the functional central limit theorem.
- Define

$$s(t, f) = \int_0^t \mathbf{1}(f(s) \geqslant 0) ds, \qquad t(s, f) = \inf(t \geqslant 0 : s(t, f) \geqslant s)$$

- Removing the negative excursions from simple random walk gives reflected random walk, so

$$\{S_n^*(t(s, S_n^*)) : s \geqslant 0\} \stackrel{d}{=} \{|S_n^*(s)| : s \geqslant 0\}.$$

$\square$

Excursions

Proof.

- Since the mapping $f \mapsto f(t(\cdot, f))$ is continuous on the part of $C[0,1]$ for which
$$\lim_{\epsilon \downarrow 0} \mathrm{meas}(s \in [0,t] : -\epsilon \leqslant f(s) \leqslant \epsilon) = 0$$
which holds for Brownian motion with probability 1, the equality in distribution for Brownian motion holds by the functional central limit theorem.

$\square$

Excursions

Proof of theorem.

Observe

$$\int_0^{\tau_a} \mathbf{1}(0 \leqslant B(s) \leqslant a)ds = \inf\{s \geqslant 0 : B(t(s)) = a\}.$$

Also,

$$\inf\{s \geqslant 0 : B(t(s)) = a\} \stackrel{d}{=} \inf\{s \geqslant 0 : |B(s)| = a\} = \sigma_a.$$

$\square$