Math 639: Lecture 20

Hausdorff dimension

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Hausdorff dimension

This lecture follows Mörters and Peres, Chapter 4.

Minkowski dimension

Definition

Suppose E is a bounded metric space with metric ρ. A *covering* of E is a finite or countable collection of sets

$$E_1, E_2, E_3, \ldots \text{ with } E \subset \bigcup_{i=1}^{\infty} E_i.$$

Define, for $\epsilon > 0$,

$$M(E, \epsilon) = \min\Big\{k \geqslant 1 : \text{there exists finite covering } E \subset \bigcup_{i=1}^{k} E_i \\ \text{with } \max_i |E_i| \leqslant \epsilon\Big\}$$

where $|A|$ is the diameter of the set A.

Minkowski dimension

Definition

The *lower Minkowski dimension* of bounded metric space E is

$$\underline{\dim}_M E := \liminf_{\epsilon \downarrow 0} \frac{\log M(E,\epsilon)}{\log \frac{1}{\epsilon}}$$

and the *upper Minkowski dimension* is

$$\overline{\dim}_M E := \limsup_{\epsilon \downarrow 0} \frac{\log M(E,\epsilon)}{\log \frac{1}{\epsilon}}.$$

When equality holds, the *Minkowski dimension* is

$$\dim_M E = \underline{\dim}_M E = \overline{\dim}_M E.$$

Minkowski dimension

Example

The Cantor set

$$C = \left\{ \sum_{i=1}^{\infty} \frac{x_i}{3^i} : x_i \in \{0, 2\} \right\} \subset [0, 1].$$

If $3^{-n+1} \geqslant \epsilon > 3^{-n}$ then C may be covered by 2^n intervals of length ϵ and not fewer than 2^{n-2} such intervals, so that the dimension is $\frac{\log 2}{\log 3}$.

Minkowski dimension

Example

Singletons have dimension 0. The set

$$E := \left\{\frac{1}{n} : n \in \mathbb{N}\right\} \cup \{0\}$$

requires a separate interval of length $\frac{1}{M}$ for every n such that $\frac{1}{n(n-1)} > \frac{1}{M}$, so that the lower dimension is at least $\frac{1}{2}$. The dimension is $\frac{1}{2}$, since the remaining part of the sequence can be covered by $O(\sqrt{M})$ such intervals.

Thus Minkowski dimension is not stable under countable union.

Hausdorff dimension

Definition

For every $\alpha \geqslant 0$ the α-*Hausdorff content* of a metric space E is defined as

$$\mathscr{H}_\infty^\alpha(E) = \inf \left\{ \sum_{i=1}^\infty |E_i|^\alpha : E \subset \bigcup_{i=1}^\infty E_i \right\}.$$

If $0 \leqslant \alpha \leqslant \beta$ and $H_\infty^\alpha(E) = 0$ then $H_\infty^\beta(E) = 0$. Define the *Hausdorff dimension* of E to be

$$\dim E = \inf \{\alpha \geqslant 0 : \mathscr{H}_\infty^\alpha(E) = 0\} = \sup \{\alpha \geqslant 0 : \mathscr{H}_\infty^\alpha(E) > 0\}.$$

Hausdorff measure

Definition

Let X be a metric space and $E \subset X$. For every $\alpha \geqslant 0$ and $\delta > 0$ define

$$\mathscr{H}_\delta^\alpha(E) = \inf\left\{\sum_{i=1}^\infty |E_i|^\alpha : E \subset \bigcup_{i=1}^\infty E_i, \sup_i |E_i| \leqslant \delta\right\}.$$

Then

$$\mathscr{H}^\alpha(E) = \sup_{\delta>0} \mathscr{H}_\delta^\alpha(E) = \lim_{\delta\downarrow 0} \mathscr{H}_\delta^\alpha(E)$$

is the α-*Hausdorff measure* of the set E.

Hausdorff measure

The α-Hausdorff measure satisfies

- $\mathscr{H}^\alpha(\varnothing) = 0$
- $\mathscr{H}^\alpha\left(\bigcup_{i=1}^\infty E_i\right) \leqslant \sum_{i=1}^\infty \mathscr{H}^\alpha(E_i)$ for any sequence $E_1, E_2, E_3, ... \subset X$
- $\mathscr{H}^\alpha(E) \leqslant \mathscr{H}^\alpha(D)$ if $E \subset D \subset X$

and thus is an outer measure.

Hausdorff measure

Proposition

For every metric space E we have

$$\mathscr{H}^\alpha(E) = 0 \;\Leftrightarrow\; \mathscr{H}^\alpha_\infty(E) = 0$$

and therefore

$$\begin{aligned}\dim E &= \inf\{\alpha : \mathscr{H}^\alpha(E) = 0\} = \inf\{\alpha : \mathscr{H}^\alpha(E) < \infty\} \\ &= \sup\{\alpha : \mathscr{H}^\alpha(E) > 0\} = \sup\{\alpha : \mathscr{H}^\alpha(E) = \infty\}.\end{aligned}$$

Hausdorff measure

Proof.

- If $\mathscr{H}^\alpha_\infty(E) = c > 0$ then $\mathscr{H}^\alpha_\delta(E) \geqslant c$ for all $\delta > 0$.
- Conversely, if $\mathscr{H}^\alpha_\infty(E) = 0$ then for every $\delta > 0$ there is a covering with sets of diameter at most $\delta^{\frac{1}{\alpha}}$.
- Letting $\delta \downarrow 0$ proves the equivalence.

□

Hölder continuity

Definition

Let $0 < \alpha \leqslant 1$. A function $f : (E_1, \rho_1) \to (E_2, \rho_2)$ between metric spaces is called α-Hölder continuous if there exists a (global) constant $C > 0$ such that

$$\rho_2(f(x), f(y)) \leqslant C\rho_1(x, y)^\alpha, \qquad \forall x, y \in E_1.$$

A constant C as above is called a *Hölder constant*.

If $f : (E_1, \rho_1) \to (E_2, \rho_2)$ is surjective and α-Hölder continuous with constant C, then for any $\beta \geqslant 0$,

$$\mathscr{H}^\beta(E_2) \leqslant C^\beta \mathscr{H}^{\alpha\beta}(E_1)$$

so $\dim(E_2) \leqslant \frac{1}{\alpha}\dim(E_1)$.

Graph and range

Definition

For a function $f : A \to \mathbb{R}^d$, for $A \subset [0, \infty)$, we define the *graph* to be

$$\mathrm{Graph}_f(A) = \{(t, f(t)) : t \in A\} \subset \mathbb{R}^{d+1},$$

and the *range* or *path* to be

$$\mathrm{Range}_f(A) = f(A) = \{f(t) : t \in A\} \subset \mathbb{R}^d.$$

Graph and range

Proposition

Suppose $f : [0,1] \to \mathbb{R}^d$ *is an* α*-Hölder continuous function. Then*

1. $\dim(\mathrm{Graph}_f[0,1]) \leqslant 1 + (1-\alpha)(d \wedge \frac{1}{\alpha})$
2. *For any* $A \subset [0,1]$*, we have* $\dim \mathrm{Range}_f(A) \leqslant \frac{\dim A}{\alpha}$.

Graph and range

Proof.

- Since f is α-Hölder continuous there is a constant C such that, if $s, t \in [0,1]$ with $|t-s| \leqslant \epsilon$, then $|f(t)-f(s)| \leqslant C\epsilon^{\alpha}$.
- Cover $[0,1]$ by no more than $\lceil \frac{1}{\epsilon} \rceil$ intervals of length ϵ. The image of each interval is contained in a ball of diameter $2C\epsilon^{\alpha}$.
- Cover each such ball by $\ll \epsilon^{d\alpha-d}$ balls of diameter ϵ. This results in a cover of the graph with $\epsilon^{d\alpha-d-1}$ products of balls and intervals, which gives part of the first bound.
- Otherwise, note that each interval of size $(\epsilon/C)^{1/\alpha}$ is mapped into a ball of radius ϵ in the range. The number of such balls required is order $\epsilon^{-1/\alpha}$, which gives the second part of the bound.
- The second part is similar.

□

Graph and range

Corollary

For any fixed set $A \subset [0, \infty)$ the graph of a d-dimensional Brownian motion satisfies, a.s.

$$\dim(\mathrm{Graph}(A)) \leqslant \begin{cases} 3/2 & d = 1 \\ 2 & d \geqslant 2 \end{cases}$$

and its range satisfies, a.s.

$$\dim \mathrm{Range}(A) \leqslant (2 \dim A) \wedge d.$$

Range of Brownian motion

Theorem

Let $\{B(t) : t \geqslant 0\}$ be a Brownian motion in dimension $d \geqslant 2$. Then almost surely, for any set $A \subset [0, \infty)$ we have

$$\mathscr{H}^2(\mathrm{Range}(A)) = 0.$$

Proof.

- Let Cube $= [0,1)^d$. It suffices to show that $\mathscr{H}^2(\text{Range}[0,\infty) \cap \text{Cube}) = 0$ for Brownian motion started at $x \notin \text{Cube}$. Also, we may assume that $d \geqslant 3$, since 2 dimensional Brownian motion is a projection, which does not increase the Hausdorff measure.
- Define the occupation measure μ by

$$\mu(A) = \int_0^\infty \mathbf{1}_A(B(s))ds, \qquad A \subset \mathbb{R}^d, \text{ Borel.}$$

- Let $\mathscr{D}_k$ be the collection of all cubes $\prod_{i=1}^d [n_i 2^{-k}, (n_i+1)2^{-k})$ where $n_1, ..., n_d \in \{0, 1, ..., 2^k - 1\}$.

□

Range of Brownian motion

Proof.

- Fix a threshold m and let $M > m$. We call $D \in \mathscr{D}_k$ with $k \geqslant m$ a *big cube* if
$$\mu(D) \geqslant \frac{1}{\epsilon} 2^{-2k}.$$
- The collection $\mathscr{C}(M)$ consists of all maximal big cubes $D \in \mathscr{D}_k$, $m \leqslant k \leqslant M$ together with those cubes $D \in \mathscr{D}_M$ which are not contained in a big cube but intersect Range$[0, \infty)$.
- The sets of $\mathscr{C}(M)$ are a cover of Range$[0, \infty) \cap$ Cube with sets of diameter at most $\sqrt{d} 2^{-m}$.

□

Proof.

- Given a cube $D \in \mathscr{D}_M$ let $D = D_M \subset D_{M-1} \subset \ldots \subset D_m$ with $D_k \in \mathscr{D}_k$ the sequence of cubes containing D. Let D_k^* be the cube with the same center as D_k and $\frac{3}{2}$ its side length.
- Let $\tau(D)$ be the first hitting time of cube D and $\tau_k = \inf\{t > \tau(D) : B(t) \notin D_k^*\}$ the first exit time from D_k^*.
- Let Child $= [0, \frac{1}{2})^d$ and define the expanded sets Cube* and Child*.
- Define $\tau = \inf\{t > 0 : B(t) \notin \text{Cube}^*\}$ and

$$q := \sup_{y \in \text{Child}^*} \text{Prob}_y \left(\int_0^\tau \mathbf{1}_{\text{Cube}}(B(s)) ds \leqslant \frac{1}{\epsilon} \right) < 1.$$

□

Range of Brownian motion

Proof.

- Using the strong Markov property

$$\begin{aligned}
&\mathrm{Prob}_x\left(\mu(D_k) \leqslant \frac{1}{\epsilon}2^{-2k}, \forall M > k \geqslant m \middle| \tau(D) < \infty\right)\\
&\leqslant \mathrm{Prob}_x\left(\int_{\tau_{k+1}}^{\tau_k} \mathbf{1}_{D_k}(B(s))ds \leqslant \frac{1}{\epsilon}2^{-2k}, M > k \geqslant m \middle| \tau(D) < \infty\right)\\
&\leqslant \prod_{k=m}^{M-1} \sup_{y \in D_{k+1}^*} \mathrm{Prob}_y\left(2^{2k}\int_0^{\tilde{\tau}_k} \mathbf{1}_{D_k}(B(s))ds \leqslant \frac{1}{\epsilon}\right) \leqslant q^{M-m}.
\end{aligned}$$

□

Proof.

- Since $\mathrm{Prob}_x(\tau(D) < \infty) \leqslant c2^{-M(d-2)}$ for a constant $c > 0$ the probability that a cube $D \in \mathscr{D}_M$ is in the cover is

$$\mathrm{Prob}_x\left(\mu(D_k) \leqslant \frac{1}{\epsilon 2^{2k}}, M > k \geqslant m, \tau(D) < \infty\right) \leqslant c2^{-M(d-2)}q^{M-m}.$$

- The 2-value of a given such cube is $d2^{-2M}$. The number of such cubes is 2^{dM}. Thus the expected contribution of all cubes in $\mathscr{C}(M) \cap \mathscr{D}_M$ is at most cdq^{M-m}.

□

Proof.

- The contribution of the remaining cubes in $\mathscr{C}(M) \cap \bigcup_{k=m}^{M-1} \mathscr{D}_k$ is bounded by

$$\begin{aligned} \sum_{k=m}^{M-1} d2^{-2k} \sum_{D \in \mathscr{C}(M) \cap \mathscr{D}_k} \mathbf{1}\left(\mu(D) \geqslant \frac{1}{\epsilon 2^{2k}}\right) &\leqslant d\epsilon \sum_{k=m}^{M-1} \sum_{D \in \mathscr{C}(M) \cap \mathscr{D}_k} \mu(D) \\ &\leqslant d\epsilon\mu(\text{Cube}). \end{aligned}$$

- Letting $\epsilon \downarrow 0$ and choosing $M = M(\epsilon)$ appropriately large, both terms are forced to 0.

□

The mass distribution principle

Definition

We call a measure μ on the Borel sets of a metric space E a *mass distribution* on E, if

$$0 < \mu(E) < \infty.$$

The mass distribution principle

Theorem

Suppose E is a metric space and $\alpha \geqslant 0$. If there is a mass distribution μ on E and constants $C > 0$ and $\delta > 0$ such that

$$\mu(V) \leqslant C|V|^{\alpha},$$

for all closed sets V with diameter $|V| \leqslant \delta$, then

$$\mathscr{H}^{\alpha}(E) \geqslant \frac{\mu(E)}{C} > 0,$$

and hence $\dim E \geqslant \alpha$.

The mass distribution principle

Proof.

Suppose that $U_1, U_2, \ldots$ is a cover of E by arbitrary sets with $|U_i| \leqslant \delta$. Let V_i be the closure of U_i and note that $|U_i| = |V_i|$. We have

$$0 < \mu(E) \leqslant \mu\left(\bigcup_{i=1}^{\infty} V_i\right) \leqslant \sum_{i=1}^{\infty} \mu(V_i) \leqslant C \sum_{i=1}^{\infty} |U_i|^{\alpha}.$$

Taking the inf and letting $\delta \downarrow 0$ gives the claim. □

Record time

Definition

Let $\{B(t) : t \geqslant 0\}$ be a linear Brownian motion and $\{M(t) : t \geqslant 0\}$ the associated maximum process. A time $t \geqslant 0$ is a *record time* for the Brownian motion if $M(t) = B(t)$ and the set of all record times for the Brownian motion is denoted by Rec.

Record time

Lemma

Almost surely, $\dim(\mathrm{Rec} \cap [0,1]) \geqslant \frac{1}{2}$ *and hence* $\dim(\mathrm{Zeros} \cap [0,1]) \geqslant \frac{1}{2}$.

Record time

Proof.

- $t \mapsto M(t)$ is continuous and increasing, hence is the distribution function of a positive measure μ, with $\mu(a,b] = M(b) - M(a)$.
- The measure μ is supported on Rec.
- For $\alpha < \frac{1}{2}$, Brownian motion is a.s. locally α-Hölder continuous
- Thus there exists a constant C_α such that, for all $a, b \in [0,1]$

$$M(b) - M(a) \leqslant \max_{0 \leqslant h \leqslant b-a} B(a+h) - B(a) \leqslant C_\alpha (b-a)^\alpha.$$

- By the mass distribution principle, a.s.

$$\dim(\mathrm{Rec} \cap [0,1]) \geqslant \alpha.$$

- The claim for Zeros follows because $Y(t) = M(t) - B(t)$ is reflected Brownian motion.

□

Zeros

Lemma

There is an absolute constant C such that, for any $a, \epsilon > 0$,

$$\mathrm{Prob}\left(\textit{there exists } t \in (a, a+\epsilon) \textit{ with } B(t) = 0\right) \leqslant C\sqrt{\frac{\epsilon}{a+\epsilon}}.$$

Zeros

Proof.

- Let $A = \{|B(a+\epsilon)| \leqslant \sqrt{\epsilon}\}$. Thus

$$\mathrm{Prob}(A) = \mathrm{Prob}\left(|B(1)| \leqslant \sqrt{\frac{\epsilon}{a+\epsilon}}\right) \leqslant 2\sqrt{\frac{\epsilon}{a+\epsilon}}.$$

- Let T be the stopping time $T = \inf\{t \geqslant a : B(t) = 0\}$

$$\begin{aligned}\mathrm{Prob}(A) &\geqslant \mathrm{Prob}\left(A \cap \{0 \in B[a, a+\epsilon]\}\right) \\ &\geqslant \mathrm{Prob}(T \leqslant a+\epsilon) \min_{a \leqslant t \leqslant a+\epsilon} \mathrm{Prob}(|B(a+\epsilon)| \leqslant \sqrt{\epsilon} | B(t) = 0).\end{aligned}$$

The minimum is achieved at $t = a$ where

$$\mathrm{Prob}(|B(a+\epsilon)| \leqslant \sqrt{\epsilon} | B(a) = 0) = \mathrm{Prob}(|B(1)| \leqslant 1)$$

which is a constant. □

Zeros

Theorem

Let $\{B(t) : 0 \leqslant t \leqslant 1\}$ *be a linear Brownian motion. Then with probability 1 we have*

$$\dim(\text{Zeros} \cap [0,1]) = \dim(\text{Rec} \cap [0,1]) = \frac{1}{2}.$$

Zeros

Proof.

- Let $Z(I)$ indicate that there is a zero in interval I. For any $\epsilon > 0$ and sufficiently large k, the previous lemma gives

$$\mathrm{E}[Z(I)] \leqslant c_1 2^{-k/2}, \qquad \forall I \in \mathscr{D}_k, I \subset (\epsilon, 1-\epsilon).$$

- Thus the covering of $\{t \in (\epsilon, 1-\epsilon) : B(t) = 0\}$ by all $I \in \mathscr{D}_k$ with $I \cap (\epsilon, 1-\epsilon) \neq \varnothing$ and $Z(I) = 1$ has expected $\frac{1}{2}$-value

$$\mathrm{E}\left[\sum_{\substack{I \in \mathscr{D}_k \\ I \cap (\epsilon, 1-\epsilon) \neq \varnothing}} Z(I) 2^{-k/2}\right] = \sum_{\substack{I \in \mathscr{D}_k \\ I \cap (\epsilon, 1-\epsilon) \neq \varnothing}} \mathrm{E}[Z(I)] 2^{-k/2} \leqslant c_1.$$

□

Zeros

Proof.

- By Fatou,

$$\mathrm{E}\left[\liminf_{k\to\infty}\sum_{\substack{I\in\mathscr{D}_k\\ I\cap(\epsilon,1-\epsilon)}} Z(I)2^{-k/2}\right] \leqslant \liminf_{k\to\infty}\mathrm{E}\left[\sum_{\substack{I\in\mathscr{D}_k\\ I\cap(\epsilon,1-\epsilon)\neq\varnothing}} Z(I)2^{-k/2}\right] \leqslant c_1.$$

- It follows that

$$\mathscr{H}^{\frac{1}{2}}\{t\in(\epsilon,1-\epsilon): B(t)=0\}<\infty.$$

Letting $\epsilon\downarrow 0$, the claim follows. □

The energy method

Definition

Suppose μ is a mass distribution on a metric space (E, ρ) and $\alpha \geqslant 0$. The *α-potential* of a point $x \in E$ with respect to μ is defined as

$$\phi_\alpha(x) = \int \frac{d\mu(y)}{\rho(x,y)^\alpha}.$$

The *α-energy* of μ is

$$I_\alpha(\mu) = \int \phi_\alpha(x) d\mu(x) = \int\int \frac{d\mu(x) d\mu(y)}{\rho(x,y)^\alpha}.$$

The energy method

Theorem (Energy method)

Let $\alpha \geqslant 0$ and μ be a mass distribution on a metric space E. Then, for every $\epsilon > 0$, we have

$$\mathscr{H}_{\epsilon}^{\alpha}(E) \geqslant \frac{\mu(E)^2}{\iint_{\rho(x,y)<\epsilon} \frac{d\mu(x)d\mu(y)}{\rho(x,y)^{\alpha}}}.$$

Hence, if $I_\alpha(\mu) < \infty$ then $\mathscr{H}^{\alpha}(E) = \infty$ and, in particular, $\dim E \geqslant \alpha$.

The energy method

Proof.

- If $\{A_n : n = 1, 2, \ldots\}$ is any disjoint covering of E with sets of diameter at most ϵ then

$$\iint_{\rho(x,y)<\epsilon} \frac{d\mu(x)d\mu(y)}{\rho(x,y)^\alpha} \geqslant \sum_{n=1}^{\infty} \iint_{A_n \times A_n} \frac{d\mu(x)d\mu(y)}{\rho(x,y)^\alpha} \geqslant \sum_{n=1}^{\infty} \frac{\mu(A_n)^2}{|A_n|^\alpha},$$

- Given $\delta > 0$ choose a covering such that, additionally,

$$\sum_{n=1}^{\infty} |A_n|^\alpha \leqslant \mathscr{H}_\epsilon^\alpha(E) + \delta.$$

□

The energy method

Proof.

- By Cauchy-Schwarz,

$$\mu(E)^2 \leqslant \left(\sum_{n=1}^{\infty} \mu(A_n)\right)^2 \leqslant \sum_{n=1}^{\infty} |A_n|^\alpha \sum_{n=1}^{\infty} \frac{\mu(A_n)^2}{|A_n|^\alpha} \leqslant (\mathscr{H}_\epsilon^\alpha(E) + \delta) \iint_{\rho(x,y)<\epsilon} \frac{d\mu(x)d\mu(y)}{\rho(x,y)^\alpha}.$$

- Letting $\delta \downarrow 0$ proves the inequality, while if $I_\alpha(\mu) < \infty$ then $\mathscr{H}_\epsilon^\alpha(E) \to 0$ as $\epsilon \to 0$.

□

The dimension of Brownian motion

Theorem (Taylor 1953)

Let $\{B(t) : 0 \leqslant t \leqslant 1\}$ *be d-dimensional Brownian motion.*

1. *If* $d = 1$, *then* $\dim \mathrm{Graph}[0,1] = \frac{3}{2}$ *a.s.*
2. *If* $d \geqslant 2$, *then* $\dim \mathrm{Range}[0,1] = \dim \mathrm{Graph}[0,1] = 2$ *a.s.*

Proof.

- For 1, let $\alpha < \frac{3}{2}$ and define a measure μ on the graph by

$$\mu(A) = \text{meas}(0 \leqslant t \leqslant 1 : (t, B(t)) \in A)$$

  for $A \subset [0,1] \times \mathbb{R}$ a Borel set.

- The α-energy of μ is

$$\iint \frac{d\mu(x)d\mu(y)}{|x-y|^\alpha} = \int_0^1 \int_0^1 \frac{dsdt}{(|t-s|^2 + |B(t)-B(s)|^2)^{\frac{\alpha}{2}}}.$$

- Thus

$$\mathrm{E}\, I_\alpha(\mu) \leqslant 2 \int_0^1 \mathrm{E}\left((t^2 + B(t)^2)^{-\frac{\alpha}{2}} \right) dt.$$

□

The dimension of Brownian motion

Proof.

- Let $p(z) = \frac{\exp(-\frac{z^2}{2})}{\sqrt{2\pi}}$. The expectation is

$$2\int_0^\infty (t^2 + tz^2)^{-\frac{\alpha}{2}} p(z)dz.$$

Split the integral at $z = \sqrt{t}$ to bound it by a constant times

$$\int_0^{\sqrt{t}} t^{-\alpha}dz + \int_{\sqrt{t}}^\infty (tz^2)^{-\alpha/2} p(z)dz = t^{\frac{1}{2}-\alpha} + t^{-\alpha/2}\int_{\sqrt{t}}^\infty z^{-\alpha}p(z)dz$$
$$\ll t^{\frac{1}{2}-\alpha} + t^{-\alpha/2} + t^{\frac{1}{2}-\alpha}.$$

- The integral over t thus converges for $\alpha < \frac{3}{2}$.

□

Proof.

- For 2, when $d \geqslant 2$, let $\alpha < 2$ and put the occupation measure on Range$[0,1]$, so

$$\mu(A) = \text{meas}(B^{-1}(A) \cap [0,1])$$

for $A \subset \mathbb{R}^d$, Borel. Thus

$$\int_{\mathbb{R}^d} f(x) d\mu(x) = \int_0^1 f(B(t)) dt.$$

□

The dimension of Brownian motion

Proof.

- We have

$$\mathrm{E}\iint \frac{d\mu(x)d\mu(y)}{|x-y|^\alpha} = \mathrm{E}\int_0^1\int_0^1 \frac{dsdt}{|B(t)-B(s)|^\alpha}$$

  and

$$\begin{aligned}\mathrm{E}\,|B(t)-B(s)|^{-\alpha} &= \mathrm{E}[(|t-s|^{\frac{1}{2}}|B(1)|)^{-\alpha}]\\ &= |t-s|^{-\alpha/2}\int_{\mathbb{R}^d}\frac{c_d e^{-\frac{|z|^2}{2}}}{|z|^\alpha}dz\\ &= c(d,\alpha)|t-s|^{-\alpha/2}.\end{aligned}$$

- Thus $\mathrm{E}\, I_\alpha(\mu) = c\int_0^1\int_0^1 \frac{dsdt}{|t-s|^{\alpha/2}} \leqslant 2c\int_0^1 \frac{du}{u^{\alpha/2}} < \infty$. The claim now follows by the energy method.

$\square$

Trees

Definition

A *tree* $T = (V, E)$ is a connected graph with finite or countable set V of *vertices*, which includes a distinguished vertex ρ designated *root*, and a set $E \subset V \times V$ of ordered *edges* such that

- For every vertex $v \in V$ the set $\{w \in V : (w, v) \in E\}$ consists of exactly one element $\overline{v}$, the *parent*, except for the *root* $\rho \in V$, which has no parent.
- For every vertex v there is a unique self-avoiding path from the root to v and the number of edges in this path is the *order* or *generation* $|v|$ of the vertex $v \in V$.
- For every $v \in V$, the set of *offspring* or *children* of $\{w \in V : (v, w) \in E\}$ is finite.

Rays

Definition

- For any $v, w \in V$ we denote $v \wedge w$ the furthest element from the root common to the paths connecting (ρ, v) and (ρ, w). Write $v \leqslant w$ if v is an *ancestor* of w, which is equivalent to $v = v \wedge w$.
- Every infinite path started in the root is called a *ray*. The set of rays is denoted ∂T and is called the *boundary* of T. Given paths ξ and η, let $\xi \wedge \eta$ be the last vertex in common, and $|\xi \wedge \eta|$ the number of edges in common. $|\xi - \eta| := 2^{-|\xi \wedge \eta|}$.
- A set Π of edges is called a *cutset* if every ray includes an edge from Π.

Flows

Definition

A *capacity* is a function $C : E \to [0, \infty)$. A *flow* of strength $c > 0$ through a tree with capacities C is a mapping $\theta : E \to [0, c]$ such that

- For the root we have $\sum_{\overline{w}=\rho} \theta(\rho, w) = c$ and for every vertex $v \neq \rho$
$$\theta(\overline{v}, v) = \sum_{w:\overline{w}=v} \theta(v, w),$$
so that the flow into and out of each vertex other than the root is conserved.
- $\theta(e) \leqslant C(e)$, i.e. the flow through the edge e is bounded by its capacity.

Max-flow min-cut theorem

Theorem (Max-flow min-cut theorem)

Let T be a tree with capacity C. Then

$$\max\{\mathrm{strength}(\theta) : \theta \text{ a flow with capacities } C\}$$
$$= \inf\left\{\sum_{e\in\Pi} C(e) : \Pi \text{ a cutset}\right\}.$$

Max-flow min-cut theorem

Proof.

- The LHS is a maximum by a diagonalization argument.
- Every infinite cutset Π contains a finite cutset $\Pi' \subset \Pi$. To see this, note that otherwise it would be possible to find an infinite sequence of rays such that the jth ray has its first j elements not in Π. An infinite ray not meeting Π is found by taking a limit.
- Let θ be a flow with capacities C and Π an arbitrary cutset. Let A be the set of vertices which are connected to ρ by a path not meeting the cutset. By the previous argument, this set is finite.

□

Max-flow min-cut theorem

Proof.

- Define

$$\phi(v,e) := \begin{cases} 1 & e = (v,w), \text{ some } w \in V \\ -1 & e = (w,v), \text{ some } w \in V \\ 0 & \text{otherwise} \end{cases}.$$

- We have

$$\begin{aligned} \text{strength}(\theta) &= \sum_{e \in E} \phi(\rho, e)\theta(e) = \sum_{v \in A} \sum_{e \in E} \phi(v,e)\theta(e) \\ &= \sum_{e \in E} \theta(e) \sum_{v \in A} \phi(v,e) \leqslant \sum_{e \in \Pi} \theta(e) \leqslant \sum_{e \in \Pi} C(e) \end{aligned}$$

□

Max-flow min-cut theorem

Proof.

- To prove the reverse inequality, let T_n denote the tree consisting of those vertices and edges at distance at most n from the root.
- Let Π be a cutset with edges in E_n
- A flow θ of strength $c > 0$ through T_n with capacities C has the condition
$$\theta(\overline{v}, v) = \sum_{w:\overline{w}=v} \theta(v, w),$$
is required for vertices $v \neq \rho$ with $|v| < n$.

□

Max-flow min-cut theorem

Proof.

- Let θ be a flow in T_n of maximal strength c with capacities C
- Call a path $\rho = v_0, v_1, ..., v_n$ an augmenting sequence if $\theta(v_i, v_{i+1}) < C(v_i, v_{i+1})$. By maximality, such an augmenting sequence does not exist.
- Since no such path exists, there is a minimal cutset Π consisting entirely of edges in E_n with $\theta(e) = C(e)$.
- We have

$$\text{strength}(\theta) = \sum_{e \in E} \theta(e) \sum_{v \in A} \phi(v, e) = \sum_{e \in \Pi} \theta(e) \geqslant \sum_{e \in \Pi} C(e).$$

- The claim in general now follows by taking a limiting such sequence θ_n, $n = 1, 2, ...$

□

Frostman's lemma

Theorem (Frostman's lemma)

If $A \subset \mathbb{R}^d$ is a closed set such that $\mathscr{H}^\alpha(A) > 0$, then there exists a Borel probability measure μ supported on A and a constant $C > 0$ such that $\mu(D) \leqslant C|D|^\alpha$ for all Borel sets D.

Frostman's lemma

Proof.

- Let $A \subset [0,1]^d$.
- A compact cube of side length s in $\mathbb{R}^d$ may be split into 2^d compact cubes of side length $s/2$.
- Create a tree with the cube $[0,1]^d$ at the root, and each vertex having 2^d edges emanating from it, leading to vertices at the 2^d sub-cubes.
- Erase edges ending in vertices associated with subcubes that do not intersect A
- Rays in ∂T correspond to sequences of nested compact cubes

□

Frostman's lemma

Proof.

- There is a canonical map $\Phi : \partial T \to A$ which maps sequences of nested cubes to their intersection.
- If $x \in A$ then there is a unique element of ∂T specified by containment at each level of the tree. Thus Φ is a bijection.
- Given edge e at level n define the capacity $C(e) = (d^{\frac{1}{2}} 2^{-n})^{\alpha}$.

□

Frostman's lemma

Proof.

- Associate to cutset Π a covering of A consisting of those cubes associated to the initial vertex of each edge in the cut-set. This indeed covers A, since any ray which ends in a point a of A passes through an edge of the cutset, so that a is contained in the associated cube.
- Thus

$$\inf\left\{\sum_{e\in\Pi} C(e) : \Pi \text{ a cutset}\right\} \geqslant \inf\left\{\sum_j |A_j|^\alpha : A \subset \bigcup_j A_j\right\}.$$

□

Frostman's lemma

Proof.

- Now we define a measure on $A \cong \partial T$.
- Given an edge e, let $T(e)$ denote the set of rays of ∂T which contain e.
- Define $\tilde{\nu}(T(e)) = \theta(e)$.
- The collection $\mathscr{C}(\partial T)$ of all sets $T(e)$, together with $\varnothing$ is a semi-algebra on ∂T since if $A, B \in \mathscr{C}(\partial T)$ then $A \cap B \in \mathscr{C}(\partial T)$, and if $A \in \mathscr{C}(\partial T)$ then A^c is a finite disjoint union of sets from $\mathscr{C}(\partial T)$.
- Since the flow through any vertex is preserved, $\tilde{\nu}$ is countably additive.
- It follows that $\tilde{\nu}$ may be extended to a measure ν on the σ-algebra generated by $\mathscr{C}(\partial T)$.

$\square$

Frostman's lemma

Proof.

- Define Borel measure $\mu = \nu \circ \Phi^{-1}$ on A. Thus if C is the cube associated to the initial vertex of edge e then $\mu(C) = \theta(e)$.
- Let D be a Borel subset of $\mathbb{R}^d$ and n is the integer such that

$$2^{-n} < |D \cap [0,1]^d| \leqslant 2^{-(n-1)}.$$

- Then $D \cap [0,1]^d$ can be covered with at most 3^d cubes from the above construction of side length 2^{-n}, or diameter $d^{\frac{1}{2}} 2^{-n}$. Thus

$$\mu(D) \leqslant d^{\frac{\alpha}{2}} 3^d 2^{-n\alpha} \leqslant d^{\frac{\alpha}{2}} 3^d |D|^\alpha$$

so that μ meets the requirements of the lemma.

□

Riesz capacity

Definition

Define the *Riesz α-capacity* of a metric space (E, ρ) as

$$\mathrm{Cap}_\alpha(E) := \sup\left\{I_\alpha(\mu)^{-1} : \mu \text{ a mass distribution on } E \text{ with } \mu(E) = 1\right\}.$$

In the case of the Euclidean space $E = \mathbb{R}^d$ with $d \geqslant 3$ and $\alpha = d - 2$ the Riesz α-capacity is also known as the *Newtonian capacity*.

Riesz capacity

Theorem

For any closed set $A \subset \mathbb{R}^d$,

$$\dim A = \sup\{\alpha : \mathrm{Cap}_\alpha(A) > 0\}.$$

Proof.

- The inequality $\dim A \geqslant \sup\{\alpha : \mathrm{Cap}_\alpha(A) > 0\}$ follows from the energy method, so it remains to prove the reverse inequality.
- Suppose $\dim A > \alpha$, so that for some $\beta > \alpha$ we have $\mathscr{H}^\beta(A) > 0$.
- By Frostman's lemma, there exists a nonzero Borel probability measure μ on A and a constant C such $\mu(D) \leqslant C|D|^\beta$
- We may assume that the support of μ has diameter less than 1.

□

Riesz capacity

Proof.

- Fix $x \in A$ and for $k \geqslant 1$ let $S_k(x) = \{y : 2^{-k} < |x-y| \leqslant 2^{1-k}\}$.
- We have

$$\int_{\mathbb{R}^d} \frac{d\mu(y)}{|x-y|^\alpha} = \sum_{k=1}^\infty \int_{S_k(x)} \frac{d\mu(y)}{|x-y|^\alpha} \leqslant \sum_{k=1}^\infty \mu(S_k(x)) 2^{k\alpha}$$

$$\leqslant C \sum_{k=1}^\infty |2^{2-k}|^\beta 2^{k\alpha} = C' \sum_{k=1}^\infty 2^{k(\alpha-\beta)}.$$

- Since $\beta > \alpha$,

$$I_\alpha(\mu) \leqslant C' \sum_{k=1}^\infty 2^{k(\alpha-\beta)} < \infty.$$

$\square$

Dimension of Brownian motion

Theorem

Let $A \subset [0, \infty)$ be a closed subset and $\{B(t) : t \geqslant 0\}$ a d-dimensional Brownian motion. Then, a.s.

$$\dim B(A) = (2 \dim A) \wedge d.$$

Dimension of Brownian motion

Proof.

- The upper bound has already been proven.
- For the lower bound let $\alpha < \dim(A) \wedge (d/2)$.
- By the previous theorem there exists a Borel probability measure μ on A such that $I_\alpha(\mu) < \infty$.

□

Dimension of Brownian motion

Proof.

- Define, for $D \subset \mathbb{R}^d$ Borel, $\tilde{\mu}(D) = \mu(\{t \geqslant 0 : B(t) \in D\})$. Thus

$$\mathrm{E}[I_{2\alpha}(\tilde{\mu})] = \mathrm{E}\left[\iint \frac{d\tilde{\mu}(x)d\tilde{\mu}(y)}{|x-y|^{2\alpha}}\right] = \mathrm{E}\left[\int_0^\infty \int_0^\infty \frac{d\mu(t)d\mu(s)}{|B(t)-B(s)|^{2\alpha}}\right]$$

- The denominator has the same distribution as $|t-s|^\alpha |Z|^{2\alpha}$.
- Since $2\alpha < d$, $\mathrm{E}[|Z|^{-2\alpha}] < \infty$. Thus

$$\mathrm{E}[I_{2\alpha}(\tilde{\mu})] = \int_0^\infty \int_0^\infty \mathrm{E}[|Z|^{-2\alpha}] \frac{d\mu(t)d\mu(s)}{|t-s|^\alpha} \leqslant \mathrm{E}[|Z|^{-2\alpha}] I_\alpha(\mu) < \infty.$$

$\tilde{\mu}$ is supported on $B(A)$, so $\dim B(A) \geqslant 2\alpha$.

□