Brownian motion

A reference for the next several lectures is the book *Brownian motion* by Mörters and Peres, CUP, 2010.
Stochastic processes

Definition

Let \((T, d)\) be a metric space.

- By a random function or process \(X = (X_t)_{t \in T}\) indexed by \(T\) we mean a collection of real valued random variables \(X_t, t \in T\).
- By the finite dimensional distributions (f.d.d.) \(X\) we mean the collection of probability measures \(\mu_{t_1, \ldots, t_n}\) on \(\mathcal{B}^n\), indexed by \(n\) and distinct \(t_1, \ldots, t_n \in T\), where

\[
\mu_{t_1, \ldots, t_n}(B) = \text{Prob}((X_{t_1}, \ldots, X_{t_n}) \in B)
\]

for any Borel subset \(B\) of \(\mathbb{R}^n\).
Finite dimensional distributions

**Definition**

A collection of finite dimensional distributions is *consistent* if for any $B_k \in \mathcal{B}$ and distinct $t_k \in T$, finite $n$, and permutation $\pi \in S_n$

$$
\mu_{t_1, \ldots, t_n}(B_1 \times \cdots \times B_n) = \mu_{t_{\pi(1)}, \ldots, t_{\pi(n)}}(B_{\pi(1)} \times \cdots \times B_{\pi(n)}),
$$

and

$$
\mu_{t_1, \ldots, t_{n-1}}(B_1 \times \cdots \times B_{n-1}) = \mu_{t_1, \ldots, t_{n-1}, t_n}(B_1 \times \cdots \times B_{n-1} \times \mathbb{R}).
$$
Definition

Let \( \mathbb{R}^T \) denote the collection of all functions \( x(t) : T \to \mathbb{R} \). A finite dimensional measurable rectangle in \( \mathbb{R}^T \) is any set of the form \( \{ x(\cdot) : x(t_i) \in B_i, i = 1, \ldots, n \} \) for a positive integer \( n \), \( B_i \in \mathcal{B} \) and \( t_i \in T \). The cylindrical \( \sigma \)-algebra, \( \mathcal{B}^T \) is the \( \sigma \)-algebra generated by the finite dimensional cylindrical rectangles.
Finite dimensional distributions

Theorem

For any consistent collection of f.d.d., there exists a probability space $(\Omega, \mathcal{F}, \text{Prob})$ and a stochastic process $\omega \mapsto \{X_t(\omega), t \in T\}$ on it, whose f.d.d. are in agreement with the given collection. Further, the restriction of the probability measure $\text{Prob}$ to the $\sigma$-algebra $\mathcal{F}^X = \sigma(X_t, t \in T)$ is uniquely determined by the specified f.d.d.
A random process $X = (X_t)_{t \in T}$ defined on probability space $(\Omega, \mathcal{A}, \text{Prob})$ is said to be *separable* if there exists a negligible set $N \subset \Omega$ and a countable set $S$ in $T$ such that, for every $\omega \notin N$, every $t \in T$, and $\epsilon > 0$,

$$X_t(\omega) \in \{X_s(\omega) : s \in S, d(s, t) < \epsilon\}.$$  

This condition is met if $(T, d)$ is separable and $X$ is almost surely continuous.
Recall that a random variable $X$ is normally distributed with mean $\mu$ and variance $\sigma^2$ if

$$\text{Prob}(X > x) = \frac{1}{\sqrt{2\pi}\sigma^2} \int_x^\infty e^{-\frac{(u-\mu)^2}{2\sigma^2}} \, du.$$
A random vector \((X_1, \ldots, X_n)\) is called a Gaussian random vector if there exists an \(n \times m\) matrix \(A\) and an \(n\)-dimensional vector \(b\) such that \(X^t = AY + b\) where \(Y\) is an \(m\)-dimensional vector with independent standard normal entries.
Paul Lévy’s construction

Definition

A real valued stochastic process \( \{B(t) : t \geq 0\} \) is called a \textit{(linear) Brownian motion} with start \( x \in \mathbb{R} \) if the following holds:

- \( B(0) = x \)
- For all times \( 0 \leq t_1 \leq t_2 \leq \ldots \leq t_n \) the increments \( B(t_n) - B(t_{n-1}), B(t_{n-1}) - B(t_{n-2}), \ldots, B(t_2) - B(t_1) \) are independent random variables.
- For all \( t \geq 0 \) and \( h > 0 \), the increments \( B(t + h) - B(t) \) are normally distributed with mean 0 and variance \( h \).
- Almost surely, \( t \mapsto B(t) \) is continuous.

If \( x = 0 \) then \( B(t) \) is \textit{standard Brownian motion}. 
Paul Lévy’s construction

Definition

We say a stochastic process \( \{X(t), t \geq 0\} \) on \((\Omega, \mathcal{A}, \text{Prob})\) has property \( X \) almost surely if there exists \( A \in \mathcal{A} \) with \( \text{Prob}(A) = 1 \) such that

\[
A \subseteq \{\omega \in \Omega : t \mapsto X(t, \omega) \text{ has property } X\}.
\]
Theorem (Wiener, 1923)

Standard Brownian motion exists.
Proof.

- We construct Brownian motion on the interval $[0, 1]$ as a random element of $C[0, 1]$, the space of continuous functions on $[0, 1]$.

- Let $\mathcal{D}_n = \left\{ \frac{k}{2^n} : 0 \leq k \leq 2^n \right\}$. We first construct the joint distribution of Brownian motion on these sets, then interpolate linearly and check that the uniform limit exists and is a Brownian motion.

- Let $\mathcal{D} = \bigcup_{n=0}^{\infty} \mathcal{D}_n$, and let $(\Omega, \mathcal{A}, \text{Prob})$ be a probability space on which a collection $\{Z_t : t \in \mathcal{D}\}$ of independent standard normals is defined.
Brownian motion

Proof.

- Define $B$ on $\mathcal{D}$ iteratively by $B(1) = Z_1$, and for $n \geq 1$ and $d \in \mathcal{D}_n \setminus \mathcal{D}_{n-1}$,

$$B(d) = \frac{B(d - 2^{-n}) + B(d + 2^{-n})}{2} + \frac{Z_d}{2^{n+1}}.$$

- We claim that this construction satisfies
  - For all $r < s < t$ in $\mathcal{D}_n$, the random variable $B(t) - B(s)$ is normally distributed with mean 0 and variance $t - s$, and is independent of $B(s) - B(r)$.
  - The vectors $\{B(d) : d \in \mathcal{D}_n\}$ and $\{Z_t : t \in \mathcal{D} \setminus \mathcal{D}_n\}$ are independent.

- The second of these properties is immediate, since $B(d)$ for $d \in \mathcal{D}_n$ is a Gaussian vector on $\{Z_s : s \in \mathcal{D}_n\}$.
Brownian motion

Proof.

To check the first property, we will show the collection of increments \( \{B(d) - B(d - 2^{-n})\} \) for \( d \in \mathcal{D}_n \setminus \{0\} \) is independent, each being a Gaussian of the correct variance.

Since this is a Gaussian vector, it suffices to check the pairwise independence of its entries.

For \( d \in \mathcal{D}_n \setminus \mathcal{D}_{n-1} \),

\[
\frac{1}{2} \left[ B(d + 2^{-n}) - B(d - 2^{-n}) \right]
\]

depends only on \((Z_t : t \in \mathcal{D}_{n-1})\), and so is independent of \( Z_d \), with variance \( 2^{-(n+1)} \). It follows that \( B(d) - B(d - 2^{-n}) \) and \( B(d + 2^{-n}) - B(d) \) are independent with mean 0 and variance \( 2^{-n} \).
Brownian motion

Proof.

- The previous arguments handles pairs $B(d) - B(d - 2^{-n})$ and $B(d + 2^{-n}) - B(d)$ for $d \in \mathcal{D}_n \setminus \mathcal{D}_{n-1}$. In all other cases, the intervals of increment are separated by some $d \in \mathcal{D}_{n-1}$.

- Let $d \in \mathcal{D}_j$ with $j$ minimal satisfying this property, so that the two intervals are contained in $[d - 2^{-j}, d]$ and $[d, d + 2^{-j}]$.

- The increments are built from the independent Gaussians $B(d) - B(d - 2^{-j})$, and $B(d + 2^{-j}) - B(d)$ using disjoint variables $(Z_t : t \in \mathcal{D}_n)$, hence they are independent.
Brownian motion

Proof.

- Define

\[
F_0(t) = \begin{cases} 
Z_1 & t = 1, \\
0 & t = 0, \\
\text{linear} & 0 < t < 1
\end{cases}
\]

and

\[
F_n(t) = \begin{cases} 
2^{-(n+1)/2} Z_t & t \in \mathcal{D}_n \setminus \mathcal{D}_{n-1} \\
0 & t \in \mathcal{D}_{n-1} \\
\text{linear interpolation} & \text{otherwise}
\end{cases}
\]

- Notice that for \( d \in \mathcal{D}_n \),

\[
B(d) = \sum_{i=0}^{n} F_i(d) = \sum_{i=0}^{\infty} F_i(d).
\]
Brownian motion

**Proof.**

- Use

\[
\text{Prob}(\|Z_d\| \geq c\sqrt{n}) \leq \exp\left(\frac{-c^2n}{2}\right),
\]

so

\[
\sum_{n=0}^{\infty} \sum_{d \in \mathcal{D}_n} \text{Prob}(\|Z_d\| \geq c\sqrt{n}) \leq \sum_{n=0}^{\infty} (2^n + 1) \exp\left(\frac{-c^2n}{2}\right).
\]

This converges for \( c > \sqrt{2 \log 2} \), so that there is \( d \in \mathcal{D}_n \) with \( |Z_d| \geq c\sqrt{n} \) only finitely often with probability 1.

- It follows that there is a random but almost surely finite \( N \), so that, for all \( n > N \),

\[
\|F_n\|_\infty < c\sqrt{n}2^{-\frac{n}{2}}.
\]
Proof.  

It follows that, almost surely,

\[ B(t) = \sum_{n=0}^{\infty} B_n(t) \]

is uniformly convergent on \([0, 1]\). Thus \(B(t)\) is almost surely continuous.

To check the finite dimensional distributions, let \(t_1 < t_2 < \cdots < t_n\) in \([0, 1]\) and let \(t_{1,k} \leq t_{2,k} \leq \cdots \leq t_{n,k}\) in \(\mathcal{D}\) with \(\lim_{k \to \infty} t_{i,k} = t_i\). By continuity,

\[ B(t_{i+1}) - B(t_i) = \lim_{k \to \infty} B(t_{i+1,k}) - B(t_{i,k}). \]
Proof.

- Note \( \lim_{k \to \infty} E[B(t_{i+1,k}) - B(t_{i,k})] = 0 \) and

\[
\lim_{k \to \infty} \text{Cov}(B(t_{i+1,k}) - B(t_{i,k}), B(t_{j+1,k}) - B(t_{j,k})) = \lim_{k \to \infty} \mathbf{1}_{i=j}(t_{i+1,k} - t_{i,k}) = \mathbf{1}_{i=j}(t_{i+1} - t_{i}).
\]

- The construction of Brownian motion on \([0, 1]\) is completed by the following proposition.
Proposition

Suppose \( \{X_n : n \in \mathbb{N}\} \) is a sequence of Gaussian random vectors, and \( \lim_n X_n = X, \) almost surely. If \( b := \lim_{n \to \infty} \mathbb{E}[X_n] \) and \( C := \lim_{n \to \infty} \text{Cov} X_n \) exist, then \( X \) is Gaussian with mean \( b \) and covariance matrix \( C \).

Proof.

The convergence guarantees that the set of affine transformations defining the Gaussian vectors converges.
To construct Brownian motion on $\mathbb{R}$, take an independent sequence $B_0, B_1, \ldots$ of Brownian motions in $C[0, 1]$ and glue them together,

$$B(t) = B_{\lfloor t \rfloor}(t - \lfloor t \rfloor) + \sum_{i=0}^{\lfloor t \rfloor - 1} B_i(1), \quad t \geq 0.$$
Lemma (Scaling invariance)

Suppose \( \{B(t) : t \geq 0\} \) is a standard Brownian motion and let \( a > 0 \). The process \( \{X(t) = \frac{1}{a} B(a^2 t) : t \geq 0\} \) is also a standard Brownian motion.
Proof.

- Continuity of paths, independence and stationarity of increments are preserved by scaling.
- Note $X(t) - X(s) = \frac{1}{a}(B(a^2 t) - B(a^2 s))$ is normal with mean 0 and variance
  
  $$\frac{1}{a^2}(a^2 t - a^2 s) = t - s.$$
Theorem (Time inversion)

Suppose \( \{B(t) : t \geq 0\} \) is a standard Brownian motion. Then \( \{X(t) : t \geq 0\} \) defined by

\[
X(t) = \begin{cases} 
0 & t = 0 \\
tB\left(\frac{1}{t}\right) & t \neq 0
\end{cases}
\]

is also a standard Brownian motion.
Proof.

- The finite-dimensional distributions \((B(t_1), \ldots, B(t_n))\) of Brownian motion are Gaussian random vectors characterized by \(E[B(t_i)] = 0\) and \(\text{Cov}(B(t_i), B(t_j)) = t_i\) for \(0 \leq t_i \leq t_j\).

- \(\{X(t) : t \geq 0\}\) is also a Gaussian process with mean 0. The covariances are given for \(t > 0\) and \(h \geq 0\) by

\[
\text{Cov}(X(t + h), X(t)) = (t + h)t \text{Cov} \left( B \left( \frac{1}{t + h} \right), B \left( \frac{1}{t} \right) \right)
\]

\[
= t(t + h) \frac{1}{t + h} = t.
\]

- It follows the law of Brownian motion agrees with

\[(X(t_1), X(t_2), \ldots, X(t_n)), \quad 0 \leq t_1 \leq t_2 \leq \cdots \leq t_n.\]
Invariance properties of Brownian motion

Proof.

- By the agreement in law,
  \[
  \lim_{t \downarrow 0, t \in \mathbb{Q}} X(t) = 0, \quad \text{a.s.}
  \]

- Thus, by continuity, \( \lim_{t \downarrow 0} X(t) = 0 \) a.s.

- This proves the a.s. continuity of \( X(t) \) on \([0, \infty)\).
Definition

The Ornstein-Uhlenbeck diffusion \( \{X(t) : t \in \mathbb{R}\} \) is defined by
\[
X(t) = e^{-t} B(e^{2t}).
\]

This process is time reversible in the sense that \( \{X(t) : t \geq 0\} \) and \( \{X(-t) : t \geq 0\} \).
Theorem (Law of large numbers)

Almost surely, \( \lim_{t \to \infty} \frac{B(t)}{t} = 0 \).

Proof.

Let \( X(t) \) be the time-reversal of \( B(t) \). The statement is equivalent to \( \lim_{t \downarrow 0} X(t) = 0 \) a.s..
Theorem

There exists a constant \( C > 0 \) such that, almost surely, for every small \( h > 0 \) and all \( 0 \leq t \leq 1 - h \),

\[
|B(t + h) - B(t)| \leq C \sqrt{h \log \frac{1}{h}}.
\]
Proof.

- Recall

\[ B(t) = \sum_{n=0}^{\infty} F_n(t) \]

where \( F_n \) is piecewise linear.

- For \( c > \sqrt{2 \log 2} \) there exists a random \( N \in \mathbb{N} \) such that, for all \( n > N \),

\[ \| F'_n \|_{\infty} \leq \frac{2 \| F_n \|_{\infty}}{2^{-n}} \leq 2c \sqrt{n} 2^{\frac{n}{2}}. \]
Modulus of continuity

**Proof.**

- By the mean value theorem, for $t, t + h \in [0, 1]$

\[
|B(t + h) - B(t)| \leq \sum_{n=0}^{\infty} |F_n(t + h) - F_n(t)|
\]

\[
\leq h \sum_{n=0}^{l} \|F'_n\|_\infty + 2 \sum_{n=l+1}^{\infty} \|F_n\|_\infty.
\]

- For $l > N$, this is bounded by

\[
h \sum_{n=0}^{N} \|F'_n\|_\infty + 2ch \sum_{n=N}^{l} \sqrt{n2^{\frac{n}{2}}} + 2c \sum_{n=l+1}^{\infty} \sqrt{n2^{-\frac{n}{2}}}.
\]
Modulus of continuity

Proof.

- Choose $h$ sufficiently small so that the first term is bounded by $\sqrt{h \log \frac{1}{h}}$, and so that $l$ defined by $2^{-l} < h \leq 2^{-l+1}$ satisfies $l > N$.

- This causes the remaining terms also to be bounded by a constant times $\sqrt{h \log \frac{1}{h}}$. 

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Modulus of continuity

**Theorem**

For every $c < \sqrt{2}$, almost surely, for every $\epsilon > 0$ there exist $0 < h < \epsilon$ and $t \in [0, 1 - h]$ with

$$|B(t + h) - B(t)| \geq c \sqrt{h \log \frac{1}{h}}.$$
Modulus of continuity

Proof.

- Let $c < \sqrt{2}$. For integers $k, n \geq 0$, define

  $$A_{k,n} = \left\{ B((k + 1)e^{-n}) - B(ke^{-n}) > c\sqrt{n}e^{-\frac{n}{2}} \right\}.$$

- We have

  $$\text{Prob}(A_{k,n}) = \text{Prob}(B(1) > c\sqrt{n}) \geq \frac{c\sqrt{n}}{c^2 n + 1} \frac{1}{\sqrt{2\pi}} e^{-\frac{c^2 n}{2}}.$$

- Using $e^n \text{Prob}(A_{k,n}) \to \infty$ as $n \to \infty$ and $1 - x \leq e^{-x}$,

  $$\text{Prob} \left( \bigcap_{0 \leq k \leq e^n - 1} A_{k,n}^c \right) = (1 - \text{Prob}(A_{0,n}))^e e^n \to 0.$$
Modulus of continuity

Theorem (Lévy’s modulus of continuity)

Almost surely,

$$\limsup_{h \downarrow 0} \sup_{0 \leq t \leq 1-h} \frac{|B(t+h) - B(t)|}{\sqrt{2h \log \frac{1}{h}}} = 1.$$
Modulus of continuity

Given natural numbers \( n, m \), define \( \Lambda_n(m) \) as the collection of intervals

\[
[(k - 1 + b)2^{-n+a}, (k + b)2^{-n+a}]
\]

for \( k \in \{1, 2, \ldots, 2^n\} \), \( a, b \in \{0, \frac{1}{m}, \ldots, \frac{m-1}{m}\} \). Set \( \Lambda(m) := \bigcup_n \Lambda_n(m) \).

**Lemma**

For any fixed \( m \) and \( c > \sqrt{2} \), almost surely, there exists \( n_0 \in \mathbb{N} \) such that, for any \( n \geq n_0 \),

\[
|B(t) - B(s)| \leq c \sqrt{(t - s) \log \frac{1}{t - s}}, \quad \forall [s, t] \in \Lambda_m(n).
\]
Modulus of continuity

Proof.
Let $X$ be standard normal. By union bound,

$$\Pr\left(\sup_{k \in \{1, \ldots, 2^n\}} \sup_{a,b \in \{0, \frac{1}{m}, \ldots, \frac{m-1}{m}\}}\left| B\left((k - 1 + b)2^{-n+a}\right) - B\left((k + b)2^{-n+a}\right)\right| > c\sqrt{2^{-n+a} \log(2^{n+a})}\right)$$

$$\leq 2^n m^2 \Pr(X > c\sqrt{\log(2^n)})$$

$$\leq \frac{m^2}{c\sqrt{\log(2^n)} \sqrt{2\pi}} \frac{1}{2^{n(1-c^2/2)}}.$$ 

The bound is summable, so that the result follows by Borel-Cantelli.
Lemma

Given \( \epsilon > 0 \) there exists \( m \in \mathbb{N} \) such that for every interval \( [s, t] \subset [0, 1] \) there exists an interval \( [s', t'] \in \Lambda(m) \) with \( |t - t'| < \epsilon(t - s) \) and \( |s - s'| < \epsilon(t - s) \).
Modulus of continuity

Proof.

- Choose $m$ sufficiently large so that $\frac{1}{m} < \frac{\epsilon}{4}$ and $2\frac{1}{m} < 1 + \frac{\epsilon}{2}$.
- Given $[s, t] \subset [0, 1]$, pick $n$ such that $2^{-n} \leq t - s < 2^{-n+1}$ and $a \in \{0, \frac{1}{m}, \ldots, \frac{m-1}{m}\}$ so that $2^{-n+a} \leq t - s < 2^{-n+a} + \frac{1}{m}$.
- Pick $k \in \{1, \ldots, 2^n\}$ such that $(k - 1)2^{-n+a} < s \leq k2^{-n+a}$ and $b \in \{0, \frac{1}{m}, \ldots, \frac{m-1}{m}\}$.
- Let $s' = (k - 1 + b)2^{-n+a}$ so that $|s - s'| \leq \frac{2^{-n+a}}{m} \leq \frac{\epsilon}{4}2^{-n+1} \leq \frac{\epsilon}{2}(t - s)$.
- Choose $t' = (k + b)2^{-n+a}$. Then $|t - t'| \leq |s - s'| + |(t - s) - (t' - s')| \leq \epsilon(t - s)$. 
Proof of Lévy’s modulus of continuity.

- Given $c > \sqrt{2}$, pick $0 < \epsilon < 1$ sufficiently small so that $	ilde{c} := c - \epsilon > \sqrt{2}$. Let $m \in \mathbb{N}$ as in the previous lemma.
- Choose $n_0 \in \mathbb{N}$ sufficiently large so that, for all $n \geq n_0$ and all intervals $[s', t'] \in \Lambda_n(m)$, almost surely

$$|B(t') - B(s')| \leq \tilde{c} \sqrt{(t' - s')} \log \frac{1}{t' - s'}.$$
Proof of Lévy’s modulus of continuity.

- Applying the previous upper bound on modulus of continuity

\[
|B(t) - B(s)| \leq |B(t) - B(t')| + |B(t') - B(s')| + |B(s') - B(s)|
\]

\[
\leq C \sqrt{|t - t'| \log \frac{1}{|t - t'|}} + \tilde{c} \sqrt{(t' - s') \log \frac{1}{t' - s'}}
\]

\[
+ C \sqrt{|s - s'| \log \frac{1}{|s - s'|}}
\]

- Taking \( \epsilon > 0 \) sufficiently small, the leading constant can be made arbitrarily close to \( c \).
Hölder continuity

Definition

A function $f : [0, \infty) \rightarrow \mathbb{R}$ is said to be locally $\alpha$-Hölder continuous at $x \geq 0$, if there exists $\epsilon > 0$ and $c > 0$ such that

$$|f(x) - f(y)| \leq c|x - y|^\alpha, \quad \forall y \in B_\epsilon(x).$$

We refer to $\alpha > 0$ as the Hölder exponent and to $c > 0$ as the Hölder constant.
Hölder continuity

**Theorem**

*If* $\alpha < \frac{1}{2}$ *then, almost surely, Brownian motion is everywhere locally* $\alpha$-*Hölder continuous.*

**Proof.**

This follows as a consequence of Lévy’s bound on modulus of continuity.
Theorem

Almost surely, for all $0 < a < b < \infty$, Brownian motion is not monotone on the interval $[a, b]$. 
Monotonicity

Proof.

- Fix an interval $[a, b]$.
- If $B(s)$ is monotone on $[a, b]$ then for each subdivision $a = a_1 < a_2 < ... < a_{n+1} = b$ into $n$ subintervals $[a_i, a_{i+1}]$, the increment $B(a_{i+1}) - B(a_i)$ has a common sign.
- By independence, this happens with probability $2 \cdot 2^{-n}$. Letting $n \to \infty$, the probability of monotonicity on $[a, b]$ is 0.
- The conclusion holds for all intervals $[a, b]$, $a < b$ simultaneously by taking a union over those intervals of rational endpoints.
Recall the Hewitt-Savage 0-1 Law.

**Theorem (Hewitt-Savage 0-1 Law)**

*If $E$ is an exchangeable event for an independent, identically distributed sequence, then $\text{Prob}(E)$ is 0 or 1.*
Deviations

Proposition

Almost surely,

\[
\limsup_{n \to \infty} \frac{B(n)}{\sqrt{n}} = \infty, \quad \liminf_{n \to \infty} \frac{B(n)}{\sqrt{n}} = -\infty.
\]
Deviations

Proof.

- By Fatou’s lemma,
  \[ \text{Prob}(B(n) > c\sqrt{n} \text{ i.o.}) \geq \limsup_{n \to \infty} \text{Prob}(B(n) > c\sqrt{n}). \]

- By scaling, the limsup is equal to \( \text{Prob}(B(1) > c) > 0). \)

- Let \( X_n = B(n) - B(n-1), \) which is an exchangeable sequence, and note
  \[ \{ B(n) > c\sqrt{n} \text{ i.o.} \} = \left\{ \sum_{j=1}^{n} X_j > c\sqrt{n} \text{ i.o.} \right\} \]

so that \( B(n) > c\sqrt{n} \text{ i.o.} \) with probability 1.
For a function $f$, define *upper* and *lower right derivatives* 

\[
D^* f(t) = \limsup_{h \downarrow 0} \frac{f(t + h) - f(t)}{h},
\]

\[
D_* f(t) = \liminf_{h \downarrow 0} \frac{f(t + h) - f(t)}{h}.
\]
Theorem

Fix $t \geq 0$. Almost surely, Brownian motion is not differentiable at $t$. Moreover, $D^* B(t) = \infty$ and $D_* B(t) = -\infty$.

Proof.

Given standard Brownian motion $B$, let $X$ be the time inversion. Then

$$D^* X(0) \geq \limsup_{n \to \infty} n(X(1/n) - X(0)) \geq \limsup_{n \to \infty} \sqrt{n}X(1/n) = \limsup_{n \to \infty} \frac{B(n)}{\sqrt{n}}.$$ 

This is infinite, and the reverse bound is similar. To obtain the bounds at $t \neq 0$, translate by $t$. \qed
Almost surely, Brownian motion is nowhere differentiable. Furthermore, almost surely, for all $t$,

$$D^* B(t) = \infty, \text{ or } D_* B(t) = -\infty.$$
Proof.

- Suppose there is $t_0 \in [0, 1]$ with

  $$\limsup_{h \downarrow 0} \frac{|B(t_0 + h) - B(t_0)|}{h} < \infty,$$

  so that there is a constant $M$ with

  $$\sup_{h \in (0,1]} \frac{|B(t_0 + h) - B(t_0)|}{h} \leq M.$$

  It suffices to prove that this holds with probability 0 for any fixed $M$. 

Proof.

- If $t_0$ is contained in $\left[\frac{k-1}{2^n}, \frac{k}{2^n}\right]$ for $n > 2$, then for all $1 \leq j \leq 2^n - k$,

$$
\left| B \left( \frac{k+j}{2^n} \right) - B \left( \frac{k+j-1}{2^n} \right) \right| \leq \left| B \left( \frac{k+j}{2^n} \right) - B(t_0) \right| + \left| B \left( \frac{k+j-1}{2^n} \right) - B(t_0) \right| \leq \frac{M(2j+1)}{2^n}.
$$

- Define

$$
\Omega_{n,k} := \left\{ \left| B \left( \frac{k+j}{2^n} \right) - B \left( \frac{k+j-1}{2^n} \right) \right| \leq \frac{M(2j+1)}{2^n}, \ j = 1, 2, 3 \right\}
$$
Proof.

By independence of increments and the scaling property,

\[ \text{Prob}(\Omega_{n,k}) \leq \text{Prob}\left(|B(1)| \leq \frac{7M}{2^n}\right)^3. \]

Thus

\[ \text{Prob}\left(\bigcup_{k=1}^{2^n-3} \Omega_{n,k}\right) \leq 2^n(7M2^{-n/2})^3 = (7M)^32^{-n/2}. \]

This is summable in \( n \), so that by Borel-Cantelli, only finitely many \( \Omega_{n,k} \) occur with probability 1.
A right-continuous function \( f : [0, t] \rightarrow \mathbb{R} \) is a function of \textit{bounded variation} if

\[
V_f^{(1)}(t) := \sum_{j=1}^{k} |f(t_j) - f(t_{j-1})| < \infty
\]

where the supremum is over all \( k \in \mathbb{N} \) and partitions

\( 0 = t_0 \leq t_1 \leq \cdots \leq t_{k-1} \leq t_k = t \). If the supremum is infinite \( f \) is said to be of \textit{unbounded variation}. 
Bounded variation

**Theorem**

Suppose that the sequence of partitions

\[ 0 = t_0^{(n)} \leq t_1^{(n)} \leq \cdots \leq t_{k(n)-1}^{(n)} \leq t_{k(n)}^{(n)} = t \]

is nested, in the sense that one point is added at each step, and the mesh

\[ \Delta(n) := \sup_{1 \leq j \leq k(n)} \{ t_j^{(n)} - t_{j-1}^{(n)} \} \]

converges to 0. Then, almost surely,

\[ \lim_{n \to \infty} \sum_{j=1}^{k(n)} (B(t_j^{(n)}) - B(t_{j-1}^{(n)}))^2 = t \]

and Brownian motion is of unbounded variation.
Bounded variation

Lemma

If \( X, Z \) are independent, symmetric random variables in \( L^2 \), then

\[
\mathbb{E}[(X + Z)^2 | X^2 + Z^2] = X^2 + Z^2.
\]

Proof.

By symmetry of \( Z \),

\[
\mathbb{E}[(X + Z)^2 | X^2 + Z^2] = \mathbb{E}[(X - Z)^2 | X^2 + Z^2].
\]

It follows that \( \mathbb{E}[XZ | X^2 + Z^2] = 0 \), which suffices.
Bounded variation

Proof of variation theorem.

To deduce the unbounded variation from the mean-square claim we use the Hölder property. Let \( \alpha \in (0, 1/2) \), and let \( n \) be such that

\[
|B(a) - B(b)| \leq |a - b|^{\alpha}
\]

for all \( a, b \in [0, t] \) with \( |a - b| \leq \Delta(n) \).

Then

\[
\sum_{j=1}^{k(n)} |B(t_j^{(n)}) - B(t_{j-1}^{(n)})| \geq \Delta(n)^{-\alpha} \sum_{j=1}^{k(n)} (B(t_j^{(n)}) - B(t_{j-1}^{(n)}))^2.
\]

Define \( X_n := \sum_{j=1}^{k(n)} \left( B\left( t_j^{(n)} \right) - B\left( t_{j-1}^{(n)} \right) \right)^2 \). Let

\( \mathcal{G}_n = \sigma(X_n, X_{n+1}, \ldots) \) and

\[
\mathcal{G}_\infty := \bigcap_{k=1}^{\infty} \mathcal{G}_k \subset \cdots \subset \mathcal{G}_{n+1} \subset \mathcal{G}_n \subset \cdots \subset \mathcal{G}_1.
\]
Proof of variation theorem.

- We show that \( \{X_n : n \in \mathbb{N}\} \) is a reverse martingale, i.e. that almost surely,
  \[
  X_n = \mathbb{E}[X_{n-1} | \mathcal{G}_n], \quad n \geq 2.
  \]

- If \( s \in (t_1, t_2) \) is the inserted point, apply the lemma to the random variables \( B(s) - B(t_1) \), \( B(t_2) - B(s) \) and \( \mathcal{F} \) the \( \sigma \)-algebra generated by \( (B(s) - B(t_1))^2 + (B(t_2) - B(s))^2 \). Thus
  \[
  \mathbb{E}[(B(t_2) - B(t_1))^2 | \mathcal{F}] = (B(s) - B(t_1))^2 + (B(t_2) - B(s))^2.
  \]

Hence
  \[
  \mathbb{E} \left[ (B(t_2) - B(t_1))^2 - (B(s) - B(t_1))^2 - (B(t_2) - B(s))^2 \bigg| \mathcal{F} \right] = 0
  \]

so \( X_n \) is a reverse martingale.
Proof of variation theorem.

- Thus \( \lim_{n \uparrow \infty} X_n = E[X_1 | \mathcal{G}_\infty] \) a.s.
- We have \( E[X_1] = t \)
- The variance is bounded by

\[
\lim \inf_{n \uparrow \infty} E[(X_n - E[X_n])^2] = \lim \inf_{n \uparrow \infty} 3 \sum_{j=1}^{k(n)} (t_{j(n)}^{(n)} - t_{j-1(n)}^{(n)})^2 
\leq 3t \lim \inf_{n \uparrow \infty} \Delta(n) = 0.
\]

- Thus the limit is \( t \) a.s.