Math 639: Lecture 17

Brownian motion

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Brownian motion

A reference for the next several lectures is the book *Brownian motion* by Mörters and Peres, CUP, 2010.

Definition

Let (T, d) be a metric space.

- By a *random function* or *process* $X = (X_t)_{t \in T}$ indexed by T we mean a collection of real valued random variables X_t, $t \in T$.
- By the *finite dimensional distributions* (f.d.d.) X we mean the collection of probability measures $\mu_{t_1,...,t_n}$ on $\mathscr{B}^n$, indexed by n and distinct $t_1, ..., t_n \in T$, where

$$\mu_{t_1,...,t_n}(B) = \mathrm{Prob}((X_{t_1}, ..., X_{t_n}) \in B)$$

for any Borel subset B of $\mathbb{R}^n$.

Finite dimensional distributions

Definition

A collection of finite dimensional distributions is *consistent* if for any $B_k \in \mathscr{B}$ and distinct $t_k \in T$, finite n, and permutation $\pi \in S_n$

$$\mu_{t_1,\ldots,t_n}(B_1 \times \cdots \times B_n) = \mu_{t_{\pi(1)},\cdots,t_{\pi(n)}}(B_{\pi(1)} \times \cdots \times B_{\pi(n)}),$$

and

$$\mu_{t_1,\ldots,t_{n-1}}(B_1 \times \cdots B_{n-1}) = \mu_{t_1,\ldots,t_{n-1},t_n}(B_1 \times \cdots \times B_{n-1} \times \mathbb{R}).$$

Finite dimensional distributions

Definition

Let $\mathbb{R}^T$ denote the collection of all functions $x(t) : T \to \mathbb{R}$. A *finite dimensional measurable rectangle* in $\mathbb{R}^T$ is any set of the form $\{x(\cdot) : x(t_i) \in B_i, i = 1, ..., n\}$ for a positive integer n, $B_i \in \mathscr{B}$ and $t_i \in T$. The *cylindrical σ-algebra*, $\mathscr{B}^T$ is the σ-algebra generated by the finite dimensional cylindrical rectangles.

Finite dimensional distributions

Theorem

For any consistent collection of f.d.d., there exists a probability space $(\Omega, \mathscr{F}, \mathrm{Prob})$ *and a stochastic process* $\omega \mapsto \{X_t(\omega), t \in T\}$ *on it, whose f.d.d. are in agreement with the given collection. Further, the restriction of the probability measure* Prob *to the* σ*-algebra* $\mathscr{F}^X = \sigma(X_t, t \in T)$ *is uniquely determined by the specified f.d.d.*

Separable processes

Definition

A random process $X = (X_t)_{t \in T}$ defined on probability space $(\Omega, \mathscr{A}, \mathrm{Prob})$ is said to be *separable* if there exists a negligible set $N \subset \Omega$ and a countable set S in T such that, for every $\omega \notin N$, every $t \in T$, and $\epsilon > 0$,

$$X_t(\omega) \in \overline{\{X_s(\omega) : s \in S, d(s,t) < \epsilon\}}.$$

This condition is met if (T, d) is separable and X is almost surely continuous.

The Gaussian

Recall that a random variable X is normally distributed with mean μ and variance σ^2 if

$$\mathrm{Prob}(X > x) = \frac{1}{\sqrt{2\pi\sigma^2}} \int_x^\infty e^{-\frac{(u-\mu)^2}{2\sigma^2}} du.$$

Gaussian random vectors

Definition

A random vector $(X_1, ..., X_n)$ is called a *Gaussian random vector* if there exists an $n \times m$ matrix A and an n-dimensional vector b such that $X^t = AY + b$ where Y is an m-dimensional vector with independent standard normal entries.

Paul Lévy's construction

Definition

A real valued stochastic process $\{B(t) : t \geqslant 0\}$ is called a *(linear) Brownian motion* with start $x \in \mathbb{R}$ if the following holds:

- $B(0) = x$
- For all times $0 \leqslant t_1 \leqslant t_2 \leqslant \ldots \leqslant t_n$ the increments $B(t_n) - B(t_{n-1})$, $B(t_{n-1}) - B(t_{n-2})$, ..., $B(t_2) - B(t_1)$ are independent random variables.
- For all $t \geqslant 0$ and $h > 0$, the increments $B(t + h) - B(t)$ are normally distributed with mean 0 and variance h.
- Almost surely, $t \mapsto B(t)$ is continuous.

If $x = 0$ then $B(t)$ is *standard Brownian motion*.

Paul Lévy's construction

Definition

We say a stochastic process $\{X(t), t \geqslant 0\}$ on $(\Omega, \mathscr{A}, \mathrm{Prob})$ *has property* $\mathfrak{X}$ *almost surely* if there exists $A \in \mathscr{A}$ with $\mathrm{Prob}(A) = 1$ such that

$$A \subset \{\omega \in \Omega : t \mapsto X(t, \omega) \text{ has property } \mathfrak{X}\}.$$

Brownian motion

Theorem (Wiener, 1923)

Standard Brownian motion exists.

Brownian motion

Proof.

- We construct Brownian motion on the interval $[0,1]$ as a random element of $C[0,1]$, the space of continuous functions on $[0,1]$.
- Let $\mathscr{D}_n = \left\{\frac{k}{2^n} : 0 \leqslant k \leqslant 2^n\right\}$. We first construct the joint distribution of Brownian motion on these sets, then interpolate linearly and check that the uniform limit exists and is a Brownian motion.
- Let $\mathscr{D} = \bigcup_{n=0}^{\infty} \mathscr{D}_n$, and let $(\Omega, \mathscr{A}, \mathrm{Prob})$ be a probability space on which a collection $\{Z_t : t \in \mathscr{D}\}$ of independent standard normals is defined.

$\square$

Brownian motion

Proof.

- Define B on $\mathscr{D}$ iteratively by $B(1) = Z_1$, and for $n \geqslant 1$ and $d \in \mathscr{D}_n \backslash \mathscr{D}_{n-1}$,

$$B(d) = \frac{B(d - 2^{-n}) + B(d + 2^{-n})}{2} + \frac{Z_d}{2^{\frac{n+1}{2}}}.$$

- We claim that this construction satisfies
  - For all $r < s < t$ in $\mathscr{D}_n$, the random variable $B(t) - B(s)$ is normally distributed with mean 0 and variance $t - s$, and is independent of $B(s) - B(r)$.
  - The vectors $\{B(d) : d \in \mathscr{D}_n\}$ and $\{Z_t : t \in \mathscr{D} \backslash \mathscr{D}_n\}$ are independent.
- The second of these properties is immediate, since $B(d)$ for $d \in \mathscr{D}_n$ is a Gaussian vector on $\{Z_s : s \in \mathscr{D}_n\}$.

$\square$

Proof.

- To check the first property, we will show the collection of increments $\{B(d) - B(d - 2^{-n})\}$ for $d \in \mathscr{D}_n \backslash \{0\}$ is independent, each being a Gaussian of the correct variance.
- Since this is a Gaussian vector, it suffices to check the pairwise independence of it's entries.
- For $d \in \mathscr{D}_n \backslash \mathscr{D}_{n-1}$,

$$\frac{1}{2}\left[B(d + 2^{-n}) - B(d - 2^{-n})\right]$$

depends only on $(Z_t : t \in \mathscr{D}_{n-1})$, and so is independent of Z_d, with variance $2^{-(n+1)}$. It follows that $B(d) - B(d - 2^{-n})$ and $B(d + 2^{-n}) - B(d)$ are independent with mean 0 and variance 2^{-n}.

$\square$

Brownian motion

Proof.

- The previous arguments handles pairs $B(d) - B(d - 2^{-n})$ and $B(d + 2^{-n}) - B(d)$ for $d \in \mathscr{D}_n \backslash \mathscr{D}_{n-1}$. In all other cases, the intervals of increment are separated by some $d \in \mathscr{D}_{n-1}$
- Let $d \in \mathscr{D}_j$ with j minimal satisfying this property, so that the two intervals are contained in $[d - 2^{-j}, d]$ and $[d, d + 2^{-j}]$.
- The increments are built from the independent Gaussians $B(d) - B(d - 2^{-j})$, and $B(d + 2^{-j}) - B(d)$ using disjoint variables $(Z_t : t \in \mathscr{D}_n)$, hence they are independent.

$\square$

Brownian motion

Proof.

- Define

$$F_0(t) = \begin{cases} Z_1 & t = 1, \\ 0 & t = 0, \\ \text{linear} & 0 < t < 1 \end{cases}$$

and

$$F_n(t) = \begin{cases} 2^{-(n+1)/2} Z_t & t \in \mathscr{D}_n \backslash \mathscr{D}_{n-1} \\ 0 & t \in \mathscr{D}_{n-1} \\ \text{linear interpolation} & \text{otherwise} \end{cases}.$$

- Notice that for $d \in \mathscr{D}_n$,

$$B(d) = \sum_{i=0}^{n} F_i(d) = \sum_{i=0}^{\infty} F_i(d).$$

□

Brownian motion

Proof.

- Use
$$\mathrm{Prob}(|Z_d| \geqslant c\sqrt{n}) \leqslant \exp\left(\frac{-c^2 n}{2}\right),$$
so
$$\sum_{n=0}^{\infty} \sum_{d \in \mathscr{D}_n} \mathrm{Prob}(|Z_d| \geqslant c\sqrt{n}) \leqslant \sum_{n=0}^{\infty} (2^n+1) \exp\left(\frac{-c^2 n}{2}\right).$$
This converges for $c > \sqrt{2\log 2}$, so that there is $d \in \mathscr{D}_n$ with $|Z_d| \geqslant c\sqrt{n}$ only finitely often with probability 1.
- It follows that there is a random but almost surely finite N, so that, for all $n > N$,
$$\|F_n\|_\infty < c\sqrt{n}2^{-\frac{n}{2}}.$$

□

Proof.

- It follows that, almost surely,

$$B(t) = \sum_{n=0}^{\infty} B_n(t)$$

is uniformly convergent on $[0, 1]$. Thus $B(t)$ is almost surely continuous.

- To check the finite dimensional distributions, let $t_1 < t_2 < \cdots < t_n$ in $[0, 1]$ and let $t_{1,k} \leqslant t_{2,k} \leqslant \cdots \leqslant t_{n,k}$ in $\mathscr{D}$ with $\lim_{k\uparrow\infty} t_{i,k} = t_i$. By continuity,

$$B(t_{i+1}) - B(t_i) = \lim_{k\uparrow\infty} B(t_{i+1,k}) - B(t_{i,k}).$$

$\square$

Proof.

- Note $\lim_{k\uparrow\infty} \mathrm{E}[B(t_{i+1,k}) - B(t_{i,k})] = 0$ and

$$\begin{aligned}&\lim_{k\uparrow\infty} \mathrm{Cov}\left(B(t_{i+1,k}) - B(t_{i,k}), B(t_{j+1,k}) - B(t_{j,k})\right)\\ &= \lim_{k\uparrow\infty} \mathbf{1}_{i=j}(t_{i+1,k} - t_{i,k}) = \mathbf{1}_{i=j}(t_{i+1} - t_i).\end{aligned}$$

- The construction of Brownian motion on $[0,1]$ is completed by the following proposition.

□

Brownian motion

Proposition

Suppose $\{X_n : n \in \mathbb{N}\}$ *is a sequence of Gaussian random vectors, and* $\lim_n X_n = X$*, almost surely. If* $b := \lim_{n\to\infty} \mathrm{E}[X_n]$ *and* $C := \lim_{n\to\infty} \operatorname{Cov} X_n$ *exist, then* X *is Gaussian with mean* b *and covariance matrix* C*.*

Proof.

The convergence guarantees that the set of affine transformations defining the Gaussian vectors converges. □

Brownian motion

To construct Brownian motion on $\mathbb{R}$, take an independent sequence $B_0, B_1, ...$ of Brownian motions in $C[0,1]$ and glue them together,

$$B(t) = B_{\lfloor t \rfloor}(t - \lfloor t \rfloor) + \sum_{i=0}^{\lfloor t \rfloor - 1} B_i(1), \qquad t \geqslant 0.$$

Invariance properties of Brownian motion

Lemma (Scaling invariance)

Suppose $\{B(t) : t \geqslant 0\}$ *is a standard Brownian motion and let* $a > 0$. *The process* $\{X(t) = \frac{1}{a}B(a^2 t) : t \geqslant 0\}$ *is also a standard Brownian motion.*

Invariance properties of Brownian motion

Proof.

- Continuity of paths, independence and stationarity of increments are preserved by scaling.
- Note $X(t) - X(s) = \frac{1}{a}(B(a^2t) - B(a^2s))$ is normal with mean 0 and variance
$$\frac{1}{a^2}(a^2t - a^2s) = t - s.$$

$\square$

Invariance properties of Brownian motion

Theorem (Time inversion)

Suppose $\{B(t) : t \geqslant 0\}$ *is a standard Brownian motion. Then* $\{X(t) : t \geqslant 0\}$ *defined by*

$$X(t) = \begin{cases} 0 & t = 0 \\ tB\left(\frac{1}{t}\right) & t \neq 0 \end{cases}$$

is also a standard Brownian motion.

Invariance properties of Brownian motion

Proof.

- The finite-dimensional distributions $(B(t_1), \dots, B(t_n))$ of Brownian motion are Gaussian random vectors characterized by $\mathrm{E}[B(t_i)] = 0$ and $\mathrm{Cov}(B(t_i), B(t_j)) = t_i$ for $0 \leqslant t_i \leqslant t_j$.
- $\{X(t) : t \geqslant 0\}$ is also a Gaussian process with mean 0. The covariances are given for $t > 0$ and $h \geqslant 0$ by

$$\begin{aligned}\mathrm{Cov}(X(t+h), X(t)) &= (t+h)t \,\mathrm{Cov}\left(B\left(\frac{1}{t+h}\right), B\left(\frac{1}{t}\right)\right) \\ &= t(t+h)\frac{1}{t+h} = t.\end{aligned}$$

- It follows the law of Brownian motion agrees with

$$(X(t_1), X(t_2), \dots, X(t_n)), \qquad 0 \leqslant t_1 \leqslant t_2 \leqslant \cdots \leqslant t_n.$$

$\square$

Invariance properties of Brownian motion

Proof.

- By the agreement in law,

$$\lim_{t\downarrow 0, t\in\mathbb{Q}} X(t) = 0, \qquad a.s.$$

- Thus, by continuity, $\lim_{t\downarrow 0} X(t) = 0$ a.s.
- This proves the a.s. continuity of $X(t)$ on $[0,\infty)$.

$\square$

Definition

The Ornstein-Uhlenbeck diffusion $\{X(t) : t \in \mathbb{R}\}$ is defined by $X(t) = e^{-t}B(e^{2t})$.

This process is time reversible in the sense that $\{X(t) : t \geqslant 0\}$ and $\{X(-t) : t \geqslant 0\}$.

Law of large numbers

Theorem (Law of large numbers)

Almost surely, $\lim_{t\to\infty} \frac{B(t)}{t} = 0$.

Proof.

Let $X(t)$ be the time-reversal of $B(t)$. The statement is equivalent to $\lim_{t\downarrow 0} X(t) = 0$ a.s.. $\square$

Modulus of continuity

Theorem

There exists a constant $C > 0$ *such that, almost surely, for every small* $h > 0$ *and all* $0 \leqslant t \leqslant 1 - h$,

$$|B(t+h) - B(t)| \leqslant C\sqrt{h \log \frac{1}{h}}.$$

Modulus of continuity

Proof.

- Recall
$$B(t) = \sum_{n=0}^{\infty} F_n(t)$$
where F_n is piecewise linear.
- For $c > \sqrt{2\log 2}$ there exists a random $N \in \mathbb{N}$ such that, for all $n > N$,
$$\|F_n'\|_\infty \leqslant \frac{2\|F_n\|_\infty}{2^{-n}} \leqslant 2c\sqrt{n}2^{\frac{n}{2}}.$$

$\square$

Modulus of continuity

Proof.

- By the mean value theorem, for $t, t+h \in [0,1]$

$$\begin{aligned} |B(t+h) - B(t)| &\leqslant \sum_{n=0}^{\infty} |F_n(t+h) - F_n(t)| \\ &\leqslant h \sum_{n=0}^{l} \left\|F_n'\right\|_\infty + 2 \sum_{n=l+1}^{\infty} \left\|F_n\right\|_\infty . \end{aligned}$$

- For $l > N$, this is bounded by

$$h \sum_{n=0}^{N} \left\|F_n'\right\|_\infty + 2ch \sum_{n=N}^{l} \sqrt{n} 2^{\frac{n}{2}} + 2c \sum_{n=l+1}^{\infty} \sqrt{n} 2^{-\frac{n}{2}}.$$

□

Modulus of continuity

Proof.

- Choose h sufficiently small so that the first term is bounded by $\sqrt{h \log \frac{1}{h}}$, and so that l defined by $2^{-l} < h \leqslant 2^{-l+1}$ satisfies $l > N$.
- This causes the remaining terms also to be bounded by a constant times $\sqrt{h \log \frac{1}{h}}$.

$\square$

Modulus of continuity

Theorem

For every $c < \sqrt{2}$*, almost surely, for every* $\epsilon > 0$ *there exist* $0 < h < \epsilon$ *and* $t \in [0, 1-h]$ *with*

$$|B(t+h) - B(t)| \geqslant c\sqrt{h \log \frac{1}{h}}.$$

Modulus of continuity

Proof.

- Let $c < \sqrt{2}$. For integers $k, n \geqslant 0$, define

$$A_{k,n} = \left\{ B((k+1)e^{-n}) - B(ke^{-n}) > c\sqrt{n}e^{-\frac{n}{2}} \right\}.$$

- We have

$$\mathrm{Prob}(A_{k,n}) = \mathrm{Prob}(B(1) > c\sqrt{n}) \geqslant \frac{c\sqrt{n}}{c^2 n + 1} \frac{1}{\sqrt{2\pi}} e^{-\frac{c^2 n}{2}}.$$

- Using $e^n \,\mathrm{Prob}(A_{k,n}) \to \infty$ as $n \to \infty$ and $1 - x \leqslant e^{-x}$,

$$\mathrm{Prob}\left(\bigcap_{0 \leqslant k \leqslant e^n - 1} A_{k,n}^c \right) \doteq (1 - \mathrm{Prob}(A_{0,n}))^{e^n} \to 0.$$

$\square$

Modulus of continuity

Theorem (Lévy's modulus of continuity)

Almost surely,

$$\limsup_{h \downarrow 0} \sup_{0 \leqslant t \leqslant 1-h} \frac{|B(t+h) - B(t)|}{\sqrt{2h \log \frac{1}{h}}} = 1.$$

Modulus of continuity

Given natural numbers n, m, define $\Lambda_n(m)$ as the collection of intervals

$$[(k-1+b)2^{-n+a}, (k+b)2^{-n+a}]$$

for $k \in \{1, 2, ..., 2^n\}$, $a, b \in \{0, \frac{1}{m}, ..., \frac{m-1}{m}\}$. Set $\Lambda(m) := \bigcup_n \Lambda_n(m)$.

Lemma

For any fixed m and $c > \sqrt{2}$, almost surely, there exists $n_0 \in \mathbb{N}$ such that, for any $n \geqslant n_0$,

$$|B(t) - B(s)| \leqslant c\sqrt{(t-s)\log\frac{1}{t-s}}, \qquad \forall\, [s,t] \in \Lambda_m(n).$$

Modulus of continuity

Proof.

Let X be standard normal. By union bound,

$$\mathrm{Prob}\Bigg(\sup_{k\in\{1,\ldots,2^n\}}\sup_{a,b\in\{0,\frac{1}{m},\ldots,\frac{m-1}{m}\}}$$
$$|B((k-1+b)2^{-n+a})-B((k+b)2^{-n+a})|>c\sqrt{2^{-n+a}\log(2^{n+a})}\Bigg)$$
$$\leqslant 2^n m^2\,\mathrm{Prob}(X>c\sqrt{\log(2^n)})$$
$$\leqslant \frac{m^2}{c\sqrt{\log(2^n)}}\frac{1}{\sqrt{2\pi}}2^{n(1-c^2/2)}.$$

The bound is summable, so that the result follows by Borel-Cantelli. □

Modulus of continuity

Lemma

Given $\epsilon > 0$ *there exists* $m \in \mathbb{N}$ *such that for every interval* $[s, t] \subset [0, 1]$ *there exists an interval* $[s', t'] \in \Lambda(m)$ *with* $|t - t'| < \epsilon(t - s)$ *and* $|s - s'| < \epsilon(t - s)$.

Modulus of continuity

Proof.

- Choose m sufficiently large so that $\frac{1}{m} < \frac{\epsilon}{4}$ and $2^{\frac{1}{m}} < 1 + \frac{\epsilon}{2}$.
- Given $[s, t] \subset [0, 1]$, pick n such that $2^{-n} \leqslant t - s < 2^{-n+1}$ and $a \in \left\{0, \frac{1}{m}, ..., \frac{m-1}{m}\right\}$ so that $2^{-n+a} \leqslant t - s < 2^{-n+a+\frac{1}{m}}$.
- Pick $k \in \{1, ..., 2^n\}$ such that $(k-1)2^{-n+a} < s \leqslant k2^{-n+a}$ and $b \in \left\{0, \frac{1}{m}, ..., \frac{m-1}{m}\right\}$.
- Let $s' = (k - 1 + b)2^{-n+a}$ so that

$$|s - s'| \leqslant \frac{2^{-n+a}}{m} \leqslant \frac{\epsilon}{4} 2^{-n+1} \leqslant \frac{\epsilon}{2}(t - s).$$

- Choose $t' = (k + b)2^{-n+a}$. Then

$$|t - t'| \leqslant |s - s'| + |(t - s) - (t' - s')| \leqslant \epsilon(t - s).$$

$\square$

Modulus of continuity

Proof of Lévy's modulus of continuity.

- Given $c > \sqrt{2}$, pick $0 < \epsilon < 1$ sufficiently small so that $\tilde{c} := c - \epsilon > \sqrt{2}$. Let $m \in \mathbb{N}$ as in the previous lemma.
- Choose $n_0 \in \mathbb{N}$ sufficiently large so that, for all $n \geqslant n_0$ and all intervals $[s', t'] \in \Lambda_n(m)$, almost surely

$$|B(t') - B(s')| \leqslant \tilde{c}\sqrt{(t' - s')\log\frac{1}{t' - s'}}.$$

□

Modulus of continuity

Proof of Lévy's modulus of continuity.

- Applying the previous upper bound on modulus of continuity

$$
\begin{aligned}
|B(t) - B(s)| &\leqslant |B(t) - B(t')| + |B(t') - B(s')| + |B(s') - B(s)| \\
&\leqslant C\sqrt{|t - t'| \log \frac{1}{|t - t'|}} + \tilde{c}\sqrt{(t' - s') \log \frac{1}{t' - s'}} \\
&\quad + C\sqrt{|s - s'| \log \frac{1}{|s - s'|}}
\end{aligned}
$$

- Taking $\epsilon > 0$ sufficiently small, the leading constant can be made arbitarily close to c.

□

Hölder continuity

Definition

A function $f : [0, \infty) \to \mathbb{R}$ is said to be *locally α-Hölder continuous at* $x \geqslant 0$, if there exists $\epsilon > 0$ and $c > 0$ such that

$$|f(x) - f(y)| \leqslant c|x - y|^{\alpha}, \qquad \forall y \in B_{\epsilon}(x).$$

We refer to $\alpha > 0$ as the *Hölder exponent* and to $c > 0$ as the *Hölder constant*.

Hölder continuity

Theorem

If $\alpha < \frac{1}{2}$ then, almost surely, Brownian motion is everywhere locally α-Hölder continuous.

Proof.

This follows as a consequence of Lévy's bound on modulus of continuity. □

Monotonicity

Theorem

Almost surely, for all $0 < a < b < \infty$, Brownian motion is not monotone on the interval $[a, b]$.

Monotonicity

Proof.

- Fix an interval $[a, b]$.
- If $B(s)$ is monotone on $[a, b]$ then for each subdivision $a = a_1 < a_2 < ... < a_{n+1} = b$ into n subintervals $[a_i, a_{i+1}]$, the increment $B(a_{i+1}) - B(a_i)$ has a common sign.
- By independence, this happens with probability $2 \cdot 2^{-n}$. Letting $n \to \infty$, the probability of monotonicity on $[a, b]$ is 0.
- The conclusion holds for all intervals $[a, b]$, $a < b$ simultaneously by taking a union over those intervals of rational endpoints.

$\square$

Recall the Hewitt-Savage 0-1 Law.

Theorem (Hewitt-Savage 0-1 Law)

If E is an exchangeable event for an independent, identically distributed sequence, then $\mathrm{Prob}(E)$ is 0 or 1.

Deviations

Proposition

Almost surely,

$$\limsup_{n\to\infty} \frac{B(n)}{\sqrt{n}} = \infty, \qquad \liminf_{n\to\infty} \frac{B(n)}{\sqrt{n}} = -\infty.$$

Deviations

Proof.

- By Fatou's lemma,

$$\mathrm{Prob}(B(n) > c\sqrt{n} \text{ i.o.}) \geqslant \limsup_{n\to\infty} \mathrm{Prob}(B(n) > c\sqrt{n}).$$

- By scaling, the limsup is equal to $\mathrm{Prob}(B(1) > c) > 0$.
- Let $X_n = B(n) - B(n-1)$, which is an exchangeable sequence, and note

$$\{B(n) > c\sqrt{n}\ i.o.\} = \left\{\sum_{j=1}^{n} X_j > c\sqrt{n}\ i.o.\right\}$$

so that $B(n) > c\sqrt{n}\ i.o.$ with probability 1.

□

Derivatives

Definition

For a function f, define *upper* and *lower right derivatives*

$$D^* f(t) = \limsup_{h \downarrow 0} \frac{f(t+h) - f(t)}{h},$$

$$D_* f(t) = \liminf_{h \downarrow 0} \frac{f(t+h) - f(t)}{h}.$$

Derivatives

Theorem

Fix $t \geqslant 0$. Almost surely, Brownian motion is not differentiable at t. Moreover, $D^* B(t) = \infty$ and $D_* B(t) = -\infty$.

Proof.

Given standard Brownian motion B, let X be the time inversion. Then

$$D^* X(0) \geqslant \limsup_{n \to \infty} n(X(1/n) - X(0)) \geqslant \limsup \sqrt{n} X(1/n) = \limsup_{n \to \infty} \frac{B(n)}{\sqrt{n}}.$$

This is infinite, and the reverse bound is similar. To obtain the bounds at $t \neq 0$, translate by t. $\square$

Derivatives

Theorem (Paley, Wiener, Zygmund, 1933)

Almost surely, Brownian motion is nowhere differentiable. Furthermore, almost surely, for all t,

$$D^* B(t) = \infty, \text{ or } D_* B(t) = -\infty.$$

Derivatives

Proof.

- Suppose there is $t_0 \in [0,1]$ with

$$\limsup_{h \downarrow 0} \frac{|B(t_0+h) - B(t_0)|}{h} < \infty,$$

so that there is a constant M with

$$\sup_{h \in (0,1]} \frac{|B(t_0+h) - B(t_0)|}{h} \leqslant M.$$

It suffices to prove that this holds with probability 0 for any fixed M.

$\square$

Derivatives

Proof.

- If t_0 is contained in $\left[\frac{k-1}{2^n}, \frac{k}{2^n}\right]$ for $n > 2$, then for all $1 \leqslant j \leqslant 2^n - k$,

$$\begin{aligned}&\left|B\left(\frac{k+j}{2^n}\right) - B\left(\frac{k+j-1}{2^n}\right)\right| \\ &\leqslant \left|B\left(\frac{k+j}{2^n}\right) - B(t_0)\right| + \left|B\left(\frac{k+j-1}{2^n}\right) - B(t_0)\right| \leqslant \frac{M(2j+1)}{2^n}.\end{aligned}$$

- Define

$$\Omega_{n,k} := \left\{\left|B\left(\frac{k+j}{2^n}\right) - B\left(\frac{k+j-1}{2^n}\right)\right| \leqslant \frac{M(2j+1)}{2^n},\ j = 1,2,3\right\}$$

□

Derivatives

Proof.

- By independence of increments and the scaling property,

$$\mathrm{Prob}(\Omega_{n,k}) \leqslant \mathrm{Prob}\left(|B(1)| \leqslant \frac{7M}{2^{\frac{n}{2}}}\right)^3.$$

- Thus

$$\mathrm{Prob}\left(\bigcup_{k=1}^{2^n-3} \Omega_{n,k}\right) \leqslant 2^n(7M2^{-n/2})^3 = (7M)^3 2^{-n/2}.$$

  This is summable in n, so that by Borel-Cantelli, only finitely many $\Omega_{n,k}$ occur with probability 1.

$\square$

Bounded variation

Definition

A right-continuous function $f : [0, t] \to \mathbb{R}$ is a function of *bounded variation* if

$$V_f^{(1)}(t) := \sum_{j=1}^{k} |f(t_j) - f(t_{j-1})| < \infty$$

where the supremum is over all $k \in \mathbb{N}$ and partitions $0 = t_0 \leqslant t_1 \leqslant \cdots \leqslant t_{k-1} \leqslant t_k = t$. If the supremum is infinite f is said to be of *unbounded variation*.

Bounded variation

Theorem

Suppose that the sequence of partitions

$$0 = t_0^{(n)} \leqslant t_1^{(n)} \leqslant \cdots \leqslant t_{k(n)-1}^{(n)} \leqslant t_{k(n)}^{(n)} = t$$

is nested, in the sense that one point is added at each step, and the mesh

$$\Delta(n) := \sup_{1 \leqslant j \leqslant k(n)} \{t_j^{(n)} - t_{j-1}^{(n)}\}$$

converges to 0. Then, almost surely,

$$\lim_{n\to\infty} \sum_{j=1}^{k(n)} (B(t_j^{(n)}) - B(t_{j-1}^{(n)}))^2 = t$$

and Brownian motion is of unbounded variation.

Lemma

If X, Z are independent, symmetric random variables in L^2, then

$$E[(X+Z)^2|X^2+Z^2] = X^2+Z^2.$$

Proof.

By symmetry of Z,

$$E[(X+Z)^2|X^2+Z^2] = E[(X-Z)^2|X^2+Z^2].$$

It follows that $E[XZ|X^2+Z^2] = 0$, which suffices. $\square$

Bounded variation

Proof of variation theorem.

- To deduce the unbounded variation from the mean-square claim we use the Hölder property. Let $\alpha \in (0, 1/2)$, and let n be such that $|B(a) - B(b)| \leqslant |a - b|^{\alpha}$ for all $a, b \in [0, t]$ with $|a - b| \leqslant \Delta(n)$.
- Then

$$\sum_{j=1}^{k(n)} |B(t_j^{(n)}) - B(t_{j-1}^{(n)})| \geqslant \Delta(n)^{-\alpha} \sum_{j=1}^{k(n)} (B(t_j^{(n)}) - B(t_{j-1}^{(n)}))^2.$$

- Define $X_n := \sum_{j=1}^{k(n)} \left(B\left(t_j^{(n)}\right) - B\left(t_{j-1}^{(n)}\right)\right)^2$. Let $\mathscr{G}_n = \sigma(X_n, X_{n+1}, ...)$ and

$$\mathscr{G}_\infty := \bigcap_{k=1}^{\infty} \mathscr{G}_k \subset \cdots \subset \mathscr{G}_{n+1} \subset \mathscr{G}_n \subset \cdots \subset \mathscr{G}_1.$$

Proof of variation theorem.

- We show that $\{X_n : n \in \mathbb{N}\}$ is a reverse martingale, i.e. that almost surely,
$$X_n = \mathrm{E}[X_{n-1}|\mathscr{G}_n], \qquad n \geqslant 2.$$
- If $s \in (t_1, t_2)$ is the inserted point, apply the lemma to the random variables $B(s) - B(t_1)$, $B(t_2) - B(s)$ and $\mathscr{F}$ the σ-algebra generated by $(B(s) - B(t_1))^2 + (B(t_2) - B(s))^2$. Thus
$$\mathrm{E}[(B(t_2) - B(t_1))^2|\mathscr{F}] = (B(s) - B(t_1))^2 + (B(t_2) - B(s))^2.$$
Hence
$$\mathrm{E}\left[(B(t_2) - B(t_1))^2 - (B(s) - B(t_1))^2 - (B(t_2) - B(s))^2 \,\middle|\, \mathscr{F}\right] = 0$$
so X_n is a reverse martingale.

□

Bounded variation

Proof of variation theorem.

- Thus $\lim_{n\uparrow\infty} X_n = \mathrm{E}[X_1|\mathscr{G}_\infty]$ a.s.
- We have $\mathrm{E}[X_1] = t$
- The variance is bounded by

$$\begin{aligned}\liminf_{n\uparrow\infty} \mathrm{E}[(X_n - \mathrm{E}[X_n])^2] &= \liminf_{n\uparrow\infty} 3\sum_{j=1}^{k(n)} (t_j^{(n)} - t_{j-1}^{(n)})^2 \\ &\leqslant 3t \liminf_{n\uparrow\infty} \Delta(n) = 0.\end{aligned}$$

- Thus the limit is t a.s.

$\square$