Math 639: Lecture 14

Ergodic theory

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Stationary sequence

Definition

A sequence $X_0, X_1, \ldots$ of random variables is *stationary* if, for each k, the shifted sequence $\{X_{n+k}, n \geqslant 0\}$ has the same distribution, that is, if for each m, $(X_0, \ldots, X_m)$ is equal in distribution to $(X_k, \ldots, X_{k+m})$.

Stationary sequence

Example

- $X_0, X_1, X_2, \ldots$ are i.i.d.
- Let X_n be a Markov chain with transition probability $p(x, A)$ and stationary probability distribution π, so $\pi(A) = \int \pi(dx)p(x, A)$. If X_0 has distribution π then $X_0, X_1, X_2, \ldots$ is stationary.
- A special case of the previous example: $S = \{0, 1\}$ and $p(x, \{1 - x\}) = 1$. The stationary distribution is $\pi(0) = \pi(1) = \frac{1}{2}$. Thus $(X_0, X_1, \ldots)$ is either $(0, 1, 0, 1, \ldots)$ or $(1, 0, 1, 0, \ldots)$ with equal probability $\frac{1}{2}$.

Stationary sequence

Example

- (Rotation of the circle) Let $\Omega = [0,1)$, $\mathscr{F}$ Borel sets and P Lebesgue measure. Set $X_n(\omega) = \omega + n\theta$ mod 1. To see this as a Markov chain, set $p(x, \{y\}) = 1$ if $y = (x+\theta)$ mod 1.
- If $X_0, X_1, \ldots$ is a stationary sequence and $g : \mathbb{R}^{\{0,1,2\ldots\}} \to \mathbb{R}$ is measurable then $Y_k = g(X_k, X_{k+1}, \ldots)$ is a stationary sequence.
- (Bernoulli shift) $\Omega = [0,1)$, $\mathscr{F}$ Borel, P Lebesgue measure. $Y_0(\omega) = \omega$ and for $n \geqslant 1$, let $Y_n(\omega) = 2Y_{n-1}(\omega)$ mod 1.

Stationary sequence

Example

- (Measure preserving map) Let $(\Omega, \mathscr{F}, P)$ be a probability space. A measurable map $\phi : \Omega \to \Omega$ is *measure preserving* if $P(\phi^{-1}A) = P(A)$ for all $A \in \mathscr{F}$. Let $\phi^n = \phi(\phi^{n-1})$ be the nth iterate, $n \geqslant 1$, where $\phi^0(\omega) = \omega$. For $X \in \mathscr{F}$, $X_n(\omega) = X(\phi^n \omega)$.

Stationary sequence

- Let $Y_0, Y_1, Y_2, \dots$ be a stationary sequence in a space $(S, \mathscr{S})$. By Kolmogorov's extension theorem there is a probability measure P on $(S^{\{0,1,2,\dots\}}, \mathscr{S}^{\{0,1,2,\dots\}})$ so that the sequence $X_n(\omega) = X(\omega_n)$ has the same distribution as $Y_0, Y_1, \dots$.
- Let ϕ be the shift operator $\phi(\omega_0, \omega_1, \dots) = (\omega_1, \omega_2, \dots)$. Then ϕ is measure preserving and $X_n(\omega) = X(\phi^n \omega)$.

Stationary sequence

Theorem

Any stationary sequence $\{X_n : n \geqslant 0\}$ can be embedded in a two-sided stationary sequence $\{Y_n : n \in \mathbb{Z}\}$.

Proof.

Define

$$\mathrm{Prob}(Y_{-m} \in A_0, \ldots, Y_n \in A_{m+n}) = \mathrm{Prob}(X_0 \in A_0, \ldots, X_{m+n} \in A_{m+n})$$

and apply Kolmogorov's extension theorem to extend Prob to a probability on $(S^{\mathbb{Z}}, \mathscr{S}^{\mathbb{Z}})$. □

Ergodicity

Definition

Let ϕ be measure preserving.

- A set $A \in \mathscr{F}$ is *invariant* if $\phi^{-1}A = A$.
- A is *almost invariant* if $\text{Prob}(A \Delta \phi^{-1}(A)) = 0$.
- The class of invariant events is a σ-field, $\mathscr{I}$.
- A measure preserving transformation on $(\Omega, \mathscr{F}, \text{Prob})$ is said to be *ergodic* if $\mathscr{I}$ is trivial, in the sense that if $A \in \mathscr{I}$ then $\text{Prob}(A) \in \{0, 1\}$.

Ergodicity

Example

Let $\{X_n\}$ be a Markov chain on countable state space S, with invariant probability measure $\pi > 0$.

- If the chain is reducible, then the various irreducible components are invariant sets with measure between 0 and 1, so the chain is not ergodic.
- If the chain is irreducible, then any invariant set is either empty or the whole space, so the chain is ergodic.

Ergodicity

Example

Consider rotation on the circle, identified with $\mathbb{R}/\mathbb{Z}$, by an angle θ.

- If $\theta = \frac{m}{n}$, $0 < m < n$ integers then the rotation is not ergodic. If B is any subset of $[0, \frac{1}{n})$ then $A = \bigcup_{k=0}^{n-1} \left(B + \frac{k}{n}\right)$ is invariant.
- If θ is irrational then the sequence is ergodic. To check this, note that $x_n = n\theta \bmod 1$. If A is an invariant set with $|A| > 0$ then, for any $\delta > 0$ we can choose interval $J = [a, b)$ with $|b - a| > 0$ such that $|A \cap J| \geqslant (1 - \delta)|J|$. By translating, $|A| \geqslant 1 - 2\delta$, so $|A| = 1$.

Ergodicity

Theorem

Let $g : \mathbb{R}^{\{0,1,\ldots\}} \to \mathbb{R}$ *be measurable. If* $X_0, X_1, \ldots$ *is an ergodic stationary sequence, then* $Y_k = g(X_k, X_{k+1}, \ldots)$ *is ergodic.*

Mean ergodic theorem

Theorem

Let U be a unitary operator on a Hilbert space $\mathscr{H}$. Let P be the orthogonal projection onto $\{\psi : \psi \in \mathscr{H}, U\psi = \psi\}$. Then, for any $f \in \mathscr{H}$,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} U^n f = Pf.$$

We will prove a vast generalization of this theorem over the next several lectures.

Mean ergodic theorem

Lemma

1. *If U is unitary, then $Uf = f$ if and only if $U^*f = f$.*
2. *For any operator on a Hilbert space $\mathscr{H}$, $(\mathrm{Ran}\, A)^\perp = \mathrm{Ker}\, A^*$.*

Proof.

To prove the first statement, since $U^*U = I$, if $Uf = f$ then $U^*Uf = f = U^*f$. Meanwhile, if $U^*f = f$ then $\langle f - Uf, f - Uf\rangle = 0$ by unitarity. The second statement is immediate. □

Mean ergodic theorem

Proof of the Mean ergodic theorem.

1. First let $f = g - Ug$. Then
$$\left\| \frac{1}{N} \sum_{n=0}^{N-1} U^n f \right\| = \left\| \frac{1}{N}(g - U^N g) \right\| \leqslant \frac{2\|g\|}{N} \to 0.$$
The same holds for $f \in \overline{\mathrm{Ran}(I - U)}$ by a limiting argument.
2. If $f \in (\mathrm{Ran}(I - U))^{\perp}$ then $U^* f = f$, so $Uf = f$, and the limit is $Pf = Uf = f$.
3. Thus the statement holds on all of $\overline{\mathrm{Ran}(I - U)} \bigoplus \mathrm{Ker}(I - U^*) = \mathscr{H}$.

□

Pointwise ergodic theorem

Theorem

Let ϕ be a measure-preserving transformation on $(\Omega, \mathscr{F}, P)$. For any $X \in L^1$,

$$\frac{1}{n}\sum_{m=0}^{n-1} X(\phi^m \omega) \to \mathrm{E}[X|\mathscr{I}]$$

a.s and in L^1.

Pointwise ergodic theorem

Lemma (Maximal ergodic lemma)

Let $X_j(\omega) = X(\phi^j\omega)$, $S_k(\omega) = X_0(\omega) + \cdots + X_{k-1}(\omega)$, *and* $M_k(\omega) = \max(0, S_1(\omega), ..., S_k(\omega))$. *Then*

$$\mathrm{E}[X\mathbf{1}(M_k > 0)] \geqslant 0.$$

Pointwise ergodic theorem

Proof of Maximal ergodic lemma.

- If $j \leqslant k$ then $M_k(\phi\omega) \geqslant S_j(\phi\omega)$, so

$$X(\omega) + M_k(\phi\omega) \geqslant X(\omega) + S_j(\phi\omega) = S_{j+1}(\omega),$$

  so $X(\omega) \geqslant S_{j+1}(\omega) - M_k(\phi\omega)$, $j = 1, 2, ..., k$.
- Trivially $X(\omega) \geqslant S_1(\omega) - M_k(\phi\omega)$ since $S_1 = X$.
- Thus

$$\begin{aligned} \mathrm{E}[X(\omega)\mathbf{1}(M_k > 0)] &\geqslant \int_{M_k>0} \max(S_1(\omega), ..., S_k(\omega)) - M_k(\phi\omega) dP \\ &= \int_{M_k>0} M_k(\omega) - M_k(\phi\omega) dP \\ &\geqslant \int M_k(\omega) - M_k(\phi\omega) dP = 0. \end{aligned}$$

□

Pointwise ergodic theorem

Proof of Pointwise ergodic theorem.

- After replacing X with $X - \mathrm{E}[X|\mathscr{I}]$ we can assume that $\mathrm{E}[X|\mathscr{I}] = 0$.
- Let $\overline{X} = \limsup \frac{S_n}{n}$ and let $\epsilon > 0$, $D = \{\omega : \overline{X}(\omega) > \epsilon\}$.
- Since $\overline{X}(\phi\omega) = \overline{X}(\omega)$, $D \in \mathscr{I}$.
- Define

$$X^*(\omega) = (X(\omega) - \epsilon)\mathbf{1}_D(\omega), \qquad S_n^*(\omega) = X^*(\omega) + \cdots + X^*(\phi^{n-1}\omega)$$

$$M_n^*(\omega) = \max(0, S_1^*(\omega), ..., S_n^*(\omega)), \qquad F_n = \{M_n^* > 0\}$$

$$F = \bigcup_n F_n = \left\{\sup_{k \geqslant 1} \frac{S_k^*}{k} > 0\right\} = D.$$

□

Pointwise ergodic theorem

Proof of Pointwise ergodic theorem.

- By the Maximal ergodic theorem $\mathrm{E}[X^* \mathbf{1}(F_n > 0)] \geqslant 0$.
- Since $\mathrm{E}[|X^*|] \leqslant \mathrm{E}[|X|] + \epsilon < \infty$, the dominated convergence theorem implies $\mathrm{E}[X^* \mathbf{1}_{F_n}] \to \mathrm{E}[X^* \mathbf{1}_F]$, so $\mathrm{E}[X^* \mathbf{1}_F] \geqslant 0$.
- Since $F = D \in \mathscr{I}$,

$$0 \leqslant \mathrm{E}[X^* \mathbf{1}_D] = \mathrm{E}[(X - \epsilon)\mathbf{1}_D] = \mathrm{E}[\mathrm{E}[X|\mathscr{I}]\mathbf{1}_D] - \epsilon P(D) = -\epsilon P(D).$$

- Thus $0 = P(D) = P(\limsup S_n/n > \epsilon)$. Replacing X with $-X$ obtains $S_n/n \to 0$ a.s.

□

Pointwise ergodic theorem

Proof of Pointwise ergodic theorem.

- To get the convergence in L^1 we truncate. Let $M > 0$,

$$X'_M(\omega) = X(\omega)\mathbf{1}(|X| \leqslant M), \qquad X''_M(\omega) = X(\omega) - X'_M(\omega).$$

- By the earlier part of the proof,

$$\frac{1}{n}\sum_{m=0}^{n-1} X'_M(\phi^m\omega) \to \mathrm{E}[X'_M|\mathscr{I}] \; a.s.$$

By bounded convergence

$$\mathrm{E}\left[\left|\frac{1}{n}\sum_{m=0}^{n-1} X'_M(\phi^m\omega) - \mathrm{E}[X'_M|\mathscr{I}]\right|\right] \to 0.$$

□

Pointwise ergodic theorem

Proof of Pointwise ergodic theorem.

- To handle X''_M, bound

$$\mathrm{E}\left[\left|\frac{1}{n}\sum_{m=0}^{n-1} X''_M(\phi^m\omega)\right|\right] \leqslant \mathrm{E}[|X''_M|].$$

Since $\mathrm{E}[|\,\mathrm{E}[X''_M|\mathscr{I}]|] \leqslant \mathrm{E}[\mathrm{E}[|X''_M||\mathscr{I}]] = \mathrm{E}[|X''_M|]$.

- It follows

$$\limsup_{n\to\infty} \mathrm{E}\left[\left|\frac{1}{n}\sum_{m=0}^{n-1} X(\phi^m\omega) - \mathrm{E}[X|\mathscr{I}]\right|\right] \leqslant 2\,\mathrm{E}[|X''_M|].$$

□

Wiener's maximal equality

Theorem

Let $X_j(\omega) = X(\phi^j\omega)$, $S_k(\omega) = X_0(\omega) + \cdots + X_{k-1}(\omega)$, $A_k(\omega) = \frac{S_k(\omega)}{k}$, *and* $D_k = \max(A_1, ..., A_k)$. *If* $\alpha > 0$, *then*

$$\mathrm{Prob}(D_k > \alpha) \leqslant \alpha^{-1}\,\mathrm{E}[|X|].$$

Proof.

Let $B = \{D_k > \alpha\}$. It follows from the Maximal ergodic lemma that

$$\mathrm{E}[|X|] \geqslant \int_B XdP \geqslant \int_B \alpha dP = \alpha\,\mathrm{Prob}(B).$$

□

Example

- (i.i.d. sequence) Since $\mathscr{I}$ is trivial, the ergodic theorem implies

$$\frac{1}{n}\sum_{m=0}^{n-1} X_m \to \mathrm{E}[X_0]$$

  a.s. and in L^1.

- (Markov chains) Let $\{X_n\}$ be an irreducible Markov chain with stationary measure $\pi > 0$. Then $\mathscr{I}$ is trivial again, so

$$\frac{1}{n}\sum_{m=0}^{n-1} f(X_m) \to \sum_x f(x)\pi(x)$$

  a.s. and in L^1.

Irrational rotations

Example

- (irrational rotations) Let $\Omega = [0,1)$, $\phi(\omega) = \omega + \theta \bmod 1$ where θ is irrational. Again $\mathscr{I}$ is trivial, so for A a Borel set,

$$\frac{1}{n}\sum_{m=0}^{n-1} \mathbf{1}(\phi^m \omega \in A) \to |A|, \ a.s.$$

Irrational rotations

Theorem

If $A = [a, b)$ is an interval then the exceptional set of rotations is empty.

Proof.

Approximate the characteristic function of the interval from above and below by trigonometric polynomials. Use that $\sum_{n=0}^{N} e(k\theta) = \frac{1-e((N+1)k\theta)}{1-e(k\theta)}$, which is bounded. □

Benford's law

Let $\theta = \log_{10} 2$ and for $1 \leqslant k \leqslant 9$, $A_k = [\log_{10} k, \log_{10}(k+1))$. By the previous result,

$$\frac{1}{n}\sum_{m=0}^{n-1} \mathbf{1}_{A_k}(\phi^m(0)) \to \log_{10} \frac{k+1}{k}.$$

This says that the first digit of the powers of 2 is asymptotically distributed according to Benford's law.

Recurrence

Theorem

Let $X_1, X_2, \ldots$ be a stationary sequence taking values in $\mathbb{R}^d$ and $S_k = X_1 + \cdots + X_k$, let $A = \{S_k \neq 0 \text{ all } k \geqslant 1\}$, and let $R_n = |\{S_1, \ldots, S_n\}|$ be the number of points visited at time n. As $n \to \infty$,

$$\frac{R_n}{n} \to \mathrm{E}[\mathbf{1}_A | \mathscr{I}] \text{ a.s.}$$

Recurrence

Proof.

- Let $X_1, X_2, \ldots$ constructed on $(\mathbb{R}^d)^{\{0,1,\ldots\}}$ with $X_n(\omega) = \omega_n$, with ϕ the shift operator.
- We have $R_n \geqslant \sum_{m=1}^n \mathbf{1}_A(\phi^m \omega)$. Thus the ergodic theorem gives

$$\liminf_{n\to\infty} \frac{R_n}{n} \geqslant \mathrm{E}[\mathbf{1}_A | \mathscr{I}], \; a.s.$$

- Let $A_k = \{S_1 \neq 0, S_2 \neq 0, \ldots, S_k \neq 0\}$. One has

$$R_n \leqslant k + \sum_{m=1}^{n-k} \mathbf{1}_{A_k}(\phi^m \omega)$$

  so $\limsup_{n\to\infty} \frac{R_n}{n} \leqslant \mathrm{E}[\mathbf{1}_{A_k}|\mathscr{I}]$. Since $\mathrm{E}[\mathbf{1}_{A_k}|\mathscr{I}] \downarrow \mathrm{E}[\mathbf{1}_A|\mathscr{I}]$, the claim follows.

□

Recurrence

Theorem

Let $X_1, X_2, \ldots$ be a stationary sequence taking values in $\mathbb{Z}$ with $\mathrm{E}[|X_i|] < \infty$. Let $S_n = X_1 + \cdots + X_n$, and let $A = \{S_1 \neq 0, S_2 \neq 0, \ldots\}$.

- *If $\mathrm{E}[X_1|\mathscr{I}] = 0$ then $\mathrm{Prob}(A) = 0$.*
- *Also, if $\mathrm{Prob}(A) = 0$ then $\mathrm{Prob}(S_n = 0 \text{ i.o.}) = 1$.*

Recurrence

Proof.

- If $\mathrm{E}[X_1|\mathscr{I}] = 0$ then the ergodic theorem implies $S_n/n \to 0$ a.s.
- For any K,
$$\limsup_{n\to\infty}\left(\max_{1\leqslant k\leqslant n}\frac{|S_k|}{n}\right) \leqslant \max_{k\geqslant K}\frac{|S_k|}{k}.$$
- This tends to 0 as $K \to \infty$, so $\frac{R_n}{n} \to 0$, and $\mathrm{Prob}(A) = 0$.
- Let $F_j = \{S_i \neq 0, \text{ for } i < j, S_j = 0\}$ and $G_{j,k} = \{S_{j+i} - S_j \neq 0 \text{ for } 1 \leqslant i < k, S_{j+k} - S_j = 0\}$.
- Since $\mathrm{Prob}(A) = 0$, $\sum \mathrm{Prob}(F_k) = 1$.

□

Recurrence

Proof.

- By stationarity, $\text{Prob}(G_{j,k}) = \text{Prob}(F_k)$. Also, for fixed j, the $G_{j,k}$ are disjoint and have union of full measure, so

$$\sum_{j,k} \text{Prob}(F_j \cap G_{j,k}) = 1.$$

- It follows that $\text{Prob}(S_n = 0 \text{ at least 2 times}) = 1$. Iterating, $\text{Prob}(S_n = 0 \text{ at least } k \text{ times}) = 1$ for all k.

$\square$

Recurrence

Theorem

Let A be a set and let $T_0 = 0$, $T_n = \inf\{m > T_{n-1} : X_m \in A\}$. If $\text{Prob}(X_n \in A \text{ at least once}) = 1$, then conditioned on $X_0 \in A$, $t_n = T_n - T_{n-1}$ is a stationary sequence with

$$\mathrm{E}[T_1 | X_0 \in A] = \frac{1}{\text{Prob}[X_0 \in A]}.$$

See Durrett pp. 340-341.

Recurrence

The result is due to Poincaré.

Theorem

Suppose $\phi : \Omega \to \Omega$ *preserves* Prob *in the sense that* $\mathrm{Prob} \circ \phi^{-1} = \mathrm{Prob}$. *Let* $T_A = \inf\{n \geqslant 1 : \phi^n(\omega) \in A\}$.

1. $T_A < \infty$ *a.s. on* A
2. $\{\phi^n(\omega) \in A \text{ i.o.}\} \supset A$
3. *If* ϕ *is ergodic and* $\mathrm{Prob}(A) > 0$, *then* $\mathrm{Prob}(\phi^n(\omega) \in A \text{ i.o.}) = 1$.

Recurrence

Proof.

1. Let $B = \{\omega \in A, T_A = \infty\}$. If $\omega \in \phi^{-m}B$ then $\phi^m(\omega) \in A$, by $\phi^n(\omega) \notin A$ for $n > m$, so the $\phi^{-m}B$ are pairwise disjoint. Since ϕ is measure preserving, $\text{Prob}(B) = 0$.
2. Since ϕ^k is measure preserving,

$$\begin{aligned} 0 &= \text{Prob}(\omega \in A, \phi^{nk}(\omega) \notin A, \text{ for all } n \geqslant 1) \\ &\geqslant \text{Prob}(\omega \in A, \phi^m(\omega) \notin A, \text{ for all } m \geqslant k). \end{aligned}$$

   This holds for all k, so the claim follows.
3. $B = \{\omega : \phi^n(\omega) \in A \text{ i.o.}\}$ is invariant and contains A, hence has probability 1.

□

The subadditive ergodic theorem

Theorem (Subadditive ergodic theorem)

Suppose $X_{m,n}$, $0 \leqslant m < n$ *satisfy*

1. $X_{0,m} + X_{m,n} \geqslant X_{0,n}$
2. $\{X_{nk,(n+1)k}, n \geqslant 1\}$ *is a stationary sequence for each* k
3. *The distribution of* $\{X_{m,m+k}, k \geqslant 1\}$ *does not depend on* m.
4. $\mathrm{E}[X_{0,1}^+] < \infty$ *and for each* n, $\mathrm{E}[X_{0,n}] \geqslant \gamma_0 n$, *where* $\gamma_0 > -\infty$.

Then

1. $\lim_{n\to\infty} \frac{1}{n} \mathrm{E}[X_{0,n}] = \inf_m \frac{1}{m} \mathrm{E}[X_{0,m}] = \gamma$.
2. $X = \lim_{n\to\infty} \frac{X_{0,n}}{n}$ *exists a.s. and in* L^1, *so* $\mathrm{E}[X] = \gamma$.
3. *If the stationary sequences in 2 above are ergodic then* $X = \gamma$ *a.s.*

Examples

Example

- (Stationary sequences) Suppose $\xi_1, \xi_2, \ldots$ is a stationary sequence with $\mathrm{E}[|\xi_k|] < \infty$, and let $X_{m,n} = \xi_{m+1} + \cdots + \xi_n$. Then $X_{0,n} = X_{0,m} + X_{m,n}$.
- (Range of a random walk) Suppose $\xi_1, \xi_2, \ldots$ is a stationary sequence and let $S_n = \xi_1 + \cdots + \xi_n$. Let $X_{m,n} = |\{S_{m+1}, \ldots, S_n\}|$. Then $X_{0,m} + X_{m,n} \geqslant X_{0,n}$.
- (Longest common subsequence) Given ergodic stationary sequences $X_1, X_2, X_3, \ldots$ and $Y_1, Y_2, Y_3, \ldots$, let $L_{m,n} = \max\{K : X_{i_k} = Y_{j_k}, 1 \leqslant k \leqslant K\}$ where $m < i_1 < i_2 < \cdots < i_K \leqslant n$ and $m < j_1 < j_2 < \cdots < j_K \leqslant n$. Then

$$L_{0,m} + L_{m,n} \geqslant L_{0,n}.$$

The subadditive ergodic theorem

Proof of the subadditive ergodic theorem.

The proof is in four steps.

- We have $\mathrm{E}[|X_{0,n}|] \leqslant Cn$. To check this, use $X_{0,m}^+ + X_{m,n}^+ \geqslant X_{0,n}^+$. Thus $\mathrm{E}[X_{0,n}^+] \leqslant n\,\mathrm{E}[X_{0,1}^+] < \infty$. Combine this with $\mathrm{E}[X_{0,n}] \geqslant \gamma_0 n$ where $\gamma_0 > -\infty$.
  Let $a_n = \mathrm{E}[X_{0,n}]$. Then $a_m + a_{n-m} \geqslant a_n$, which implies
  $$\frac{a_n}{n} \to \inf_{m \geqslant 1} \frac{a_m}{m} = \gamma.$$

□

The subadditive ergodic theorem

Proof of the subadditive ergodic theorem.

- Write $n = km + \ell$. Then

$$\frac{X_{0,n}}{n} \leqslant \frac{k}{km+\ell} \frac{X_{0,m} + \cdots + X_{(k-1)m,km}}{k} + \frac{X_{km,n}}{n}.$$

The pointwise ergodic theorem gives

$$\frac{X_{0,m} + \cdots + X_{(k-1)m,km}}{k} \to A_m \text{ a.s. and in } L^1$$

where $A_m = \mathrm{E}[X_{0,m}|\mathscr{I}_m]$, and $\mathscr{I}_m$ is shift invariant for $X_{(k-1)m,km}$, $k \geqslant 1$. For fixed ℓ, $\epsilon > 0$, since $\mathrm{E}[X_{0,\ell}^+] < \infty$,

$$\sum_{k=1}^{\infty} \mathrm{Prob}(X_{km,km+\ell} > (km+\ell)\epsilon) \leqslant \sum_{k=1}^{\infty} \mathrm{Prob}(X_{0,\ell} > k\epsilon) < \infty.$$

□

The subadditive ergodic theorem

Proof of the subadditive ergodic theorem.

Combining these observations,

$$\overline{X} = \limsup \frac{X_{0,n}}{n} \leqslant \frac{A_m}{m},$$

so $E[\overline{X}] \leqslant \frac{1}{m} E[X_{0,m}]$, which implies $E[\overline{X}] \leqslant \gamma$. If the sequences are ergodic, then $\overline{X} \leqslant \gamma$.

□

The subadditive ergodic theorem

Proof of the subadditive ergodic theorem.

- Let $\underline{X} = \liminf_{n\to\infty} \frac{X_{0,n}}{n}$. Let $\epsilon > 0$ and let $Z = \epsilon + (\underline{X} \vee -M)$. Since $\mathrm{E}[\overline{X}] < \infty$, $\mathrm{E}[|Z|] < \infty$.
  Define $Y_{m,n} = X_{m,n} - (n-m)Z$ and $\underline{Y} = \liminf_{n\to\infty} \frac{Y_{0,n}}{n} \leqslant -\epsilon$.
  Define $T_m = \min\{n \geqslant 1 : Y_{m,m+n} \leqslant 0\}$.
  By stationarity, T_m is equal in distribution to T_0, so

$$\mathrm{E}[Y_{m,m+1}\mathbf{1}(T_m > N)] = \mathrm{E}[Y_{0,1}\mathbf{1}(T_0 > N)].$$

  Since $\mathrm{Prob}(T_0 < \infty) = 1$, pick N large enough so that

$$\mathrm{E}[Y_{0,1}\mathbf{1}(T_0 > N)] \leqslant \epsilon.$$

□

The subadditive ergodic theorem

Proof of the subadditive ergodic theorem.

Define

$$S_m = \begin{cases} T_m & \text{on } \{T_m \leqslant N\} \\ 1 & \text{on } \{T_m > N\} \end{cases}$$

$$\xi_m = \begin{cases} 0 & \text{on } \{T_m \leqslant N\} \\ Y_{m,m+1} & \text{on } \{T_m > N\} \end{cases} .$$

Since $Y_{m,m+T_m} \leqslant 0$ and $S_m = 1$, $Y_{m,m+1} > 0$ on $\{T_m > N\}$ we have $Y_{m,m+S_m} \leqslant \xi_m$ and $\xi_m \geqslant 0$.

□

The subadditive ergodic theorem

Proof of the subadditive ergodic theorem.

Let $R_0 = 0$ and $R_k = R_{k-1} + S_{R_{k-1}}$. Define $K = \max\{k : R_k \leqslant n\}$. We have

$$Y_{0,n} \leqslant Y_{R_0,R_1} + Y_{R_1,R_2} + \ldots + Y_{R_{K-1},R_K} + Y_{R_K,n} \leqslant \sum_{m=0}^{n-1} \xi_m + \sum_{j=1}^{N} |Y_{n-j,n-j+1}|.$$

Hence,

$$\limsup_{n\to\infty} \frac{1}{n} \mathrm{E}[Y_{0,n}] \leqslant \mathrm{E}[\xi_0] \leqslant \mathrm{E}[Y_{0,1}\mathbf{1}(T_0 > N)] \leqslant \epsilon.$$

Thus $\gamma = \lim_{n\to\infty} \frac{1}{n} \mathrm{E}[X_{0,n}] \leqslant 2\epsilon + \mathrm{E}[\underline{X} \vee -M]$. Thus $\gamma = \mathrm{E}[\underline{X}] = \mathrm{E}[\overline{X}]$ and $\underline{X} = \overline{X}$ almost surely. □

The subadditive ergodic theorem

Proof of the subadditive ergodic theorem.

See Durrett p. 346 for the convergence in L^1. □

Products of random matrices

Example

- (Products of random matrices) Suppose $A_1, A_2, \ldots$ is a stationary sequence of $k \times k$ matrices with positive entries, and let

$$\alpha_{m,n}(i,j) = (A_{m+1} \cdots A_n)(i,j)$$

Note $\alpha_{0,m}(1,1)\alpha_{m,n}(1,1) \leqslant \alpha_{0,n}(1,1)$. Set $X_{m,n} = -\log \alpha_{m,n}(1,1)$ so $X_{0,m} + X_{m,n} \geqslant X_{0,n}$. Subject to

$$\mathrm{E}[|\log A_m(i,j)|] < \infty, \text{ all } i,j$$

we obtain $\frac{1}{n}X_{0,n} \to X$ a.s.

First-passage percolation

Example

(First passage percolation) Consider $\mathbb{Z}^2$ as a graph with edges connecting $x, y \in \mathbb{Z}^2$ when $|x - y| = 1$. Assign i.i.d. non-negative edge weights $\tau(e)$ of finite mean.

- If $x_0 = x, x_1, x_2, \ldots, x_n = y$ is a path from x to y with $|x_m - x_{m-1}| = 1$, define the *travel time* to be

$$\tau(x_0, x_1) + \cdots + \tau(x_{n-1}, x_n).$$

- Define the *passage time* $t(x, y)$ to be the infimum of travel times over all paths from x to y.
- Define $X_{m,n} = t((m, 0), (n, 0))$. Since $X_{0,m} + X_{m,n} \geqslant X_{0,n}$, one obtains $\frac{X_{0,n}}{n} \to X$ a.s. We have X is almost surely constant, since it is measurable in the tail sigma field.