PROBABILITY I, SPRING 2017, HW2

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Problem 1. (Monte Carlo integration)

(1) Let f be a measurable function on $[0,1]$ with $\int_0^1 |f(x)|dx < \infty$. Let $U_1, U_2, ...$ be independent and uniformly distributed on $[0,1]$, and let

$$I_n = \frac{1}{n}\left(f(U_1) + \cdots + f(U_n)\right).$$

Show that $I_n \to I = \int_0^1 f dx$ in probability.

(2) Suppose $\int_0^1 |f(x)|^2 dx < \infty$. Use Chebyshev's inequality to estimate $\mathrm{Prob}\left(|I_n - I| > \frac{a}{n^{\frac{1}{2}}}\right)$.

Problem 2. Let $X_1, X_2, ...$ be i.i.d. with distribution satisfying $\mathrm{Prob}(X_i > x) = \frac{e}{x \log x}$ for $x \geqslant e$. Show that $\mathbf{E}[|X_i|] = \infty$, but there is a sequence of constants $\mu_n \to \infty$ so that $\frac{S_n}{n} - \mu_n$ tends to 0 in probability.

Problem 3. Let A_n be a sequence of independent events with $\mathrm{Prob}(A_n) < 1$ for all n. Show that $\mathrm{Prob}\left(\bigcup_n A_n\right) = 1$ implies $\mathrm{Prob}\left(A_n \text{ i.o.}\right) = 1$.

Problem 4. Suppose $\sum \mathrm{Prob}(A_k) = \infty$. Show that if

$$\limsup_{n\to\infty} \left(\sum_{k=1}^n \mathrm{Prob}(A_k)\right)^2 \Big/ \left(\sum_{1\leqslant j,k\leqslant n} \mathrm{Prob}(A_j \cap A_k)\right) = \alpha > 0$$

then $\mathrm{Prob}(A_n \text{ i.o.}) \geqslant \alpha$. (Hint: Use Problem 2 of HW#1 and Fatou's lemma.)

Problem 5. Let $X_1, X_2, ...$ be i.i.d. Poisson with mean 1, and let $S_n = X_1 + \cdots + X_n$. Find $\lim_{n\to\infty} \frac{1}{n} \log \mathrm{Prob}(S_n \geqslant na)$ for $a > 1$.